Mathematical Reviews

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HISTORY, BIOGRAPHY

★**Dantzig, Tobias. The bequest of the Greeks.** Charles Scribner's Sons, New York, 1955. 191 pp. $3.95.
"... a study of problems, principles and procedures which modern mathematics has inherited from Greek antiquity."

Stamatis, Evangelos. A contribution to the investigation of the geometrical algebra of the Pythagoreans. Prakt. Akad. Athēnōn **30** (1955), 262–282. (Greek. English summary)

Toth, I. Mathematicians of the Middle Ages in central Asia. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **9** (1955), no. 3, 74–83. (Romanian)

Dugas, René. Sur le paradoxe de d'Alembert. C. R. Acad. Sci. Paris **241** (1955), 1437–1438.

Kratzer, A. Fünfzig Jahre Relativitätstheorie. Math.-Phys. Semesterber. **4** (1955), 171–182.

Boyer, Carl B. Post-Cartesian analytic geometry. Scripta Math. **21** (1955), 101–135.

Finikov, S. P. On the scientific trend of the department of differential geometry of the Moscow State University. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **9** (1955), no. 3, 5–21. (Romanian)
Translation of Uspehi Mat. Nauk **9** (1954), no. 4(62), 3–18; MR **16**, 433.

★**Gorelik, G. S. Life and work of A. A. Andronov.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 3–19 (3 plates). Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

★**Aizerman, M. A. Survey of A. A. Andronov's work in the field of automatic control.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 20–42. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.
A list of Andronov's published papers is included.

Kakutani, Shizuo. Obituary: Hirotada Anzai (1919-1955). Osaka Math. J. **7** (1955), no. 1, i–ii (1 plate).

Neugebauer, O. Apollonius' planetary theory. Comm. Pure Appl. Math. **8** (1955), 641–648.

Adams, C. R., and Neugebauer, Otto. Obituary: Raymond Clare Archibald. Amer. Math. Monthly **62** (1955), 743–745.

Thorndike, Lynn. Giovanni Bianchini's astronomical instrument. Scripta Math. **21** (1955), 136–137.

★**Sadykov, H. U. Biruni i ego raboty po astronomii i matematičeskoĭ geografii. [Biruni and his work on astronomy and mathematical geography].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 152 pp. 4.40 rubles.

Taylor, Geoffrey. George Boole, 1815-1864. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 66–73.

Rhees, Rush, editor. George Boole as student and teacher. By some of his friends and pupils. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 74–78.

Hackett, Felix E. The method of George Boole. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 79–87.

Thomas, Ivo. Boole's concept of science. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 88–96.

Feys, Robert. Boole as a logician. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 97–106.

Rosser, J. Barkley. Boole and the concept of a function. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 117–120.

Eichler, Martin. Nekrolog: Heinrich Brandt. Math. Nachr. **13** (1955), 321–326.
A list of Brandt's scientific work is included.

★**Dini, Ulisse. Opere. Vol. III. Equazioni differenziali ordinarie e alle derivate parziali.** Edizioni Cremonese, Roma, 1955. 661 pp. 6000 Lire.
This volume completes the planned edition of Dini's mathematical work [for vols. I and II see MR **15**, 383; **16**, 1]. This volumes contains Dini's papers on ordinary and partial differential equations, the former with an introduction by G. Sansone, the latter with one by M. Picone.

Lévy, Paul. W. Doeblin (V. Doblin) (1915-1940). Rev. Hist. Sci. Appl. **8** (1955), 107–115.
It is a pity that this tribute to Doeblin's genius was written without reference to later developments of his work. After all there can be no greater testimony to a man's work than its influence on others. Fortunately, for Doeblin, this influence has been visible and is still continuing. On limit theorems his work has been complemented and completed by Gnedenko and other Russian authors. On Markov processes it has been carried on mostly in the United States by Doob, T. E. Harris and the reviewer. Here his mine of ideas and techniques is still being explored. The last sentence in Section 3 of the paper under review is erroneous. Doeblin did not make this assertion; it was attributed to Kolmogorov and was found to be in error by the reviewer [see p. 411 of Trans. Amer. Math. Soc. **76** (1954), 397–419; MR **16**, 149].
K. L. Chung (Syracuse, N.Y.).

Julia, Gaston. Notice nécrologique sur Henri Dulac. C. R. Acad. Sci. Paris **241** (1955), 913–916.

de Rham, Georges. Nécrologie: Gustave Dumas. Elem. Math. **10** (1955), 121–122.

Lemaitre, G. L'oeuvre scientifique d'Albert Einstein. Rev. Questions Sci. (5) **16** (1955), 475–487.

Kovan'ko, A. S. Vladimir Semënovič Fëdorov. (On the fortieth anniversary of his scientific-pedagogical activity.) Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 193–196 (1 plate). (Russian)
A list of Fëdorov's mathematical papers is included.

Costabel, Pierre. Autour de la méthode de Galilée pour la détermination des centres de gravité. Rev. Hist. Sci. Appl. **8** (1955), 116–128.

★ **Dunnington, G. Waldo. Carl Friedrich Gauss: titan of science. A study of his life and work.** Exposition Press, New York, 1955. xvi+479 pp. (33 plates) $6.00.

Toth, Imre. Gauss and non-Euclidean geometry. Gaz. Mat. Fiz. Ser. A. **7** (1955), 447–456. (Romanian)

Schoenberg, Erich, und Perlick, Alfons. Unbekannte Briefe von C. F. Gauss und Fr. W. Bessel. Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. **71** (1955), 58 pp. (2 plates).

Zühlke, P. Walter Lietzmann zum 75. Geburtstage. Math.-Phys. Semesterber. **4** (1955), 161–164 (1 plate).

Mihoc, G. Andrei Andreevič Markov. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **9** (1955), no. 2, 95–106. (Romanian)
Brief description of A. A. Markov's (1856–1922) mathematical work.

Tenca, Luigi. Antonio Nardi matematico aretino. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 491–506.

★ **Levi, Beppo. L'opera matematica di Giuseppe Peano.** In memoria di Giuseppe Peano, pp. 9–21. Liceo Scientifico Statale, Cuneo, 1955.

★ **Ascoli, Guido. I motivi fondamentali dell'opera di Giuseppe Peano.** In memoria di Giuseppe Peano, pp. 23–30. Liceo Scientifico Statale, Cuneo, 1955.

★ **Segre, Beniamino. Peano ed il Bourbakismo.** In memoria di Giuseppe Peano, pp. 31–39. Liceo Scientifico Statale, Cuneo, 1955.

★ **Boggio, Tommaso. Il calcolo geometrico di Peano.** In memoria di Giuseppe Peano, pp. 65–69. Liceo Scientifico Statale, Cuneo, 1955.

★ **Cassina, Ugo. Sul "formulario mathematico" di Peano.** In memoria di Giuseppe Peano, pp. 71–102. Liceo Scientifico Statale, Cuneo, 1955.

★ **Carruccio, Ettore. Spunti di storia delle matematiche e della logica nell'opera di G. Peano.** In memoria di Giuseppe Peano, pp. 103–114. Liceo Scientifico Statale, Cuneo, 1955.

Martín Jadraque, Valentín. Gabrio Piola (1794-1850). Gac. Mat., Madrid (1) **7** (1955), 57–59 (1 plate). (Spanish)

Gomes, Ruy Luís. On the first centenary of the birth of Henri Poincaré. Gaz. Mat., Lisboa **15** (1955), no. 60–61, 1–3 (1 plate). (Portuguese)

Hermes, Hans. Heinrich Scholz zum 70. Geburtstage. Math.-Phys. Semesterber. **4** (1955), 165–170 (1 plate).

ALGEBRA

★ **van der Waerden, B. L. Algebra. 3te Aufl.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. XXXIV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+224 pp. DM 29.60.
"Fast ungeänderte Neuauflage" [1st ed., 1931; 2d ed., 1940; MR 2, 120]. Misprints and two minor errors have been corrected and the title modified.

Lambek, J., and Moser, L. Some associative operations on integers. Math. Mag. **29** (1955), 59–62.

See also Pickert, p. 399.

Linear Algebra, Polynomials, Invariants

Gáspár, Julius. Eine neue Definition der Determinanten. Publ. Math. Debrecen **3** (1954), 257–260 (1955).
For matrices with elements in a commutative ring of characteristic zero and without zero divisors, it is shown that a non-constant function φ on square matrices to elements of the ring is the determinant if it satisfies: $\varphi(AB)=\varphi(A)\varphi(B)$, $\varphi(aA)=a^n\varphi(A)$ and $\varphi(A+B)=\sum\varphi(C)$, where the summation is over the 2^n matrices obtained by choosing for each row either the corresponding row of A or of B. *W. Givens* (Knoxville, Tenn.).

Rešetnyak, Yu. G. New proof of a theorem of N. G. Čebotarëv. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 155–157. (Russian)
The theorem in question is the following: None of the minors of the $p\times p$ matrix (a_{ij}), $a_{ij}=\varepsilon^{(i-1)(j-1)}$, $\varepsilon=\exp 2\pi i/p$, p prime, is singular [N. G. Čebotarëv, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 4(26), 3–66); MR **10**, 174]. *J. L. Brenner* (Pullman, Wash.).

Mirsky, L. An inequality for positive definite matrices. Amer. Math. Monthly **62** (1955), 428–430.
For A_i hermitian positive definite matrices and α_i positive numbers with $\sum\alpha_i=1$, $\det(\sum\alpha_iA_i)\geqq\prod_i(\det A_i)^{\alpha_i}$ with equality if and only if all A_i are equal. The proof is reduced by induction to the case of two matrices where it follows by reduction to diagonal form and use of $\alpha\lambda+1-\alpha\geqq\lambda^\alpha$ $(0<\alpha<1;\ \lambda>0)$. *W. Givens.*

Bellman, Richard. Notes on matrix theory. IV. An inequality due to Bergström. Amer. Math. Monthly 62 (1955), 172–173.

For a positive definite matrix A, $(x, Ax)(y, A^{-1}y) \geqq (x,y)^2$ for all x and y, and for fixed y the equality may be attained, so that $(y, A^{-1}y)^{-1} = \min_x (x, Ax)(x, y)^{-2}$. Taking $y_i = 1$, $y_j = 0$ for $j \neq i$ yields Bergström's result that $\varphi(A+B) \geqq \varphi(A) + \varphi(B)$ for $\varphi(A) = (\det A)(\det A_{ii})^{-1}$ with A_{ii} obtained by deleting the ith row and column of A [Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, Tanum, Oslo, 1952, pp. 264–267; MR 14, 716].

W. Givens (Knoxville, Tenn.).

Bellman, Richard, Glicksberg, Irving, and Gross, Oliver. Notes on matrix theory. VI. Amer. Math. Monthly 62 (1955), 571–572.

For $|A_k| = \lambda_1\lambda_2\cdots\lambda_k$, where $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ are the eigenvalues of the positive definite matrix A, the maximum of $\int_L \exp[-(x, Ax)]dV_k$ for all k-dimensional subspaces L is attained when L is the subspace spanned by the first k eigenvectors and its value is $[\pi^k |A|_k^{-1}]^{\frac{1}{2}}$.

W. Givens (Knoxville, Tenn.).

Bellman, Richard. Notes on matrix theory. VII. Amer. Math. Monthly 62 (1955), 647–648.

A known inequality on the products of characteristic values of symmetric matrices A, B and $\lambda A + (1-\lambda)B$ is used to derive, by a limiting process, another known inequality on sums. *W. Givens* (Knoxville, Tenn.).

Bodewig, E. Zum Matrizenkalkul. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indagationes Math. 17 (1955), 95–106; corrigenda, 260.

The calculus to which the title refers is a systematic use of the usual basis matrices E_{ik} to deal effectively with the elements of matrices while retaining throughout a matrix form for the equations. The technique is illustrated by deriving both standard results and new or extended formulas, the latter mainly for the variation in some function of a matrix. Among the results are several capable of useful mechanization on a digital computer. With A^{-1} denoted by R, some of these are: $(A+dA)^{-1} = R - RQ + RQ^2 - \cdots$ for $Q = (dA)R$; $dx = R[dv - (dA)x]$ where $Ax = v$; if ε is a bound for the uncertainty in each element of A, then $\varepsilon sz'$ is a bound for the elements of dA, where s is the sum of the column vectors of $\bar{R} = (|r_{ij}|)$ and z' is the sum of the row vectors of $\bar{R}$; if B has rank one, so $B = sz'$ for suitable vectors s and z', $(A+B)^{-1} = R + cRBR$ where $c = (1+z'Rs)^{-1}$ and this is applied to correct A^{-1} when an element of A is corrected by an amount not limited to be small. When N has k non-zero rows or columns, recursion formulas requiring $2k(n^2+n)$ multiplications are given for calculating $(A+N)^{-1}$ when A^{-1} is known. For $a = \det A$, $da = a(\text{trace } P)$, where $P = RdA$ and $dP = -P^2$ is used to get a recursion formula for $d^i a$. The corrigenda clarify a definition of two vectors (p. 102).

W. Givens (Knoxville, Tenn.).

Hsu, Pao-Lu. On a kind of transformations of matrices. Acta Math. Sinica 5 (1955), 333–346. (Chinese. English summary)

All matrices considered in this paper are square matrices with complex elements. For any matrix X, let $\bar{X}$ denote the matrix whose elements are the complex conjugates of the corresponding elements of X. Let

$$\varphi(X) = \begin{pmatrix} 0 & X \\ \bar{X} & 0 \end{pmatrix}, \qquad \psi(X) = \begin{pmatrix} 0 & X \\ -\bar{X} & 0 \end{pmatrix}.$$

Two matrices A, B are said to be $\mathfrak{A}$-similar, if there exists a non-singular P such that $B = PA\bar{P}^{-1}$. It is shown that the $\mathfrak{A}$-similarity of A, B is equivalent to the usual similarity of $\varphi(A)$ and $\varphi(B)$, also to the similarity of $\psi(A)$ and $\psi(B)$. In case of non-singular A, B, it is also equivalent to the similarity of $A\bar{A}$ and $B\bar{B}$. A canonical form for $\mathfrak{A}$-similar matrices is given. Every matrix is $\mathfrak{A}$-similar to some real matrix. A matrix can be expressed in the form $A\bar{A}$, if and only if it is similar to the square of a real matrix. Necessary and sufficient conditions in terms of elementary divisors are obtained for a matrix to be similar to a matrix of the form $\varphi(A)$ or of the form $\psi(A)$. In proving these results, the author first treats separately the case of non-singular matrices and the case of matrices X with nilpotent $X\bar{X}$. Then he passes to the general case by using the following lemma: Every matrix is $\mathfrak{A}$-similar to a direct sum $B \oplus C$, where B is non-singular and $C\bar{C}$ is nilpotent. *Ky Fan* (Notre Dame, Ind.).

Goddard, L. S., and Schneider, H. Pairs of matrices with a non-zero commutator. Proc. Cambridge Philos. Soc. 51 (1955), 551–553.

The following theorem is proved: let A, B, X and K be matrices of orders $n \times n$, $m \times m$, $n \times m$ and $m \times n$, respectively, with elements in a field F and suppose $AX = XB$, X of rank r and $f(x, y)$ any polynomial in non-commuting indeterminates and with coefficients in F. Then

$$|\lambda I_n - f(A, XK)| = \theta(\lambda)\phi(\lambda) \text{ and } |\lambda I_m - f(B, KX)| = \theta(\lambda)q(\lambda),$$

where $\theta(\lambda)$, $\phi(\lambda)$ and $q(\lambda)$ are polynomials in $F[\lambda]$ of degrees r, $n-r$ and $m-r$, respectively. Also, $\phi(\lambda)$ and $q(\lambda)$ are independent of K and divide $f(A, 0)$ and $f(B, 0)$ respectively, Moreover,

$$|\lambda I_n - f(A, XK)| \cdot |\lambda I_m - f(B, 0)| = |\lambda I_m - f(B, KX)| \cdot |\lambda I_n - f(A, 0)|.$$

The proof follows straightforwardly by an equivalence reduction of X and induced similarities on A and B.

Define: M and N have property P_r if

$$|\lambda I_n - f(M, N)| = \psi(\lambda) \prod_{i=1}^{r} (\lambda - f(\mu_i, \nu_i))$$

for all f and a fixed ordering of the roots μ_i and ν_i (in the closure of F) of M and N, thus generalizing property $P(=P_n)$ of matrices. Then, A and XK have property P_{n-r} and B and KX have property P_{m-r}. If A and XK have property P then so do B and KX, and conversely. The special cases $f(x, y) = x$, $f(x, y) = y$, $f(x, y) = x + y$ and $r = m$ with X consisting of characteristic column vectors of A, $f(x, y)$ with zero constant term and $m = n$ for $B = 0$, and $f(x, y) = x - y$ with X of full rank and $\lambda = 0$ each yield a main result of a recent paper in the literature.

W. Givens (Knoxville, Tenn.).

Dionísio, J. Joaquim. Two notes on matrices. Portugal. Math. 13 (1954), 141–144.

In the first note it is observed that a linear transformation A is necessarily scalar if it commutes with B and C, where $Bx_i = \beta_i x_i$ for x_i a basis set of vectors with $\beta_i \neq \beta_j$ for $i \neq j$ and C having a simple eigenvalue with vector $x = \sum \gamma_i x_i$ and $\gamma_i \neq 0$ for $i = 1, \cdots, n$.

In II, for singular linear transformations A and D satisfying $D^p \neq 0$ and rank $A \leqq n$-rank D^p for an integer p, it is proved that for arbitrary B there exists a matrix A' similar to A such that the elementary divisors of $A'B$ with non-zero roots are contained among those of $A'B + D^p$

(and are identical if D is nilpotent), and the characteristic equation of $A'B$ is $\lambda^{n-s}\varphi(\lambda)$ and of $A'B+D^s$ is $\varphi(\lambda)\psi(\lambda)$. *W. Givens* (Knoxville, Tenn.).

Smiley, M. F. A remark on matric polynomials and similarity of matrices. Amer. Math. Monthly **62** (1955), 173–174.

For matrix polynomials $A(x)$ and $B(x)$ in a commuting indeterminate and $B_R(C)=\sum B_iC^i$ the usual result of right substitution for x, $(A(x)B(x))_R(C)=\sum A_iB_R(C)C^i$ is immediate yet easily yields the implication

$$P(x)(Ix-M)Q(x)=Ix-N\Rightarrow N=P^{-1}MP, \text{ for } P=Q_R(N),$$

where $P(x)$ and $Q(x)$ have matrix polynomial inverses and M and N are constant. The proof holds for matrices with elements in a division ring F and a modification to σ-similarity ($N=P^{-1}MP^{\sigma}$, σ an automorphism of F) is indicated. A computational procedure for similarity reduction to canonical forms is sketched. *W. Givens.*

Afriat, S. N. Symmetric matrices, quadratic forms and linear constraints. Publ. Math. Debrecen **3** (1954), 305–308 (1955).

Concise proofs are given of the classical condition that a real symmetric matrix A be positive definite; and of the condition of H. B. Mann [Amer. Math. Monthly **50** (1943), 430–433; MR **5**, 30] that A be positive definite subject to linear constraints:

$$x\neq 0,\ U'x=0\Rightarrow x'Ax>0.$$

J. L. Brenner (Pullman, Wash.).

Cantoni, Lionello. Serie di potenze e logaritmo di una matrice. Boll. Un. Mat. Ital. (3) **10** (1955), 376–381.

This paper deals with some simple results concerning square matrices over the complex field. Let A be such a matrix and let $f(z)$ be a power series in z such that $f(A)$ is convergent. The cononical form of $f(A)$ is determined. The conditions on A that some power of A can be diagonalised are found. For a given A the conditions on $f(z)$ such that $f(A)$ can be diagonalised are also obtained.

The second section concerns lg (A), the principal value of the logarithm of a nonsingular matrix A [S. Cherubino, Rend. Mat. e Appl. (5) **14** (1954), 221–238; MR **16**, 665]. It is shown that S^{-1} lg $(A)S=$ lg $(S^{-1}AS)$ and that lg A is a unique polynomial in A of degree exactly $\mu-1$, where μ is the degree of the minimal polynomial of A.

D. E. Rutherford (St. Andrews).

Amitsur, S. A. Identities and generators of matrix rings. Bull. Res. Council Israel. Sect. A. **5** (1955), 5–10.

This paper continues the author's investigation of polynomial identities in semi-simple, finite-dimensional algebras. Let F be an infinite field, F_n the total matric algebra of dimension n^2 over F, M_n the set of polynomial identities satisfied by F_n, $P_n(x, y)=S_{n+1}(y, yx, \cdots, yx^n)$, where S_{n+1} is a standard polynomial [Amitsur and Levitzki, Proc. Amer. Math. Soc. **1** (1950), 449–463; MR **12**, 155]. Principal results: (1) If $B\in F_{n+1}$ and A_i $(i=1, \cdots, r)$ generate F_{n+1}, then $p(A_1, \cdots, A_r)=B$ for some $p\in M_n$. (2) The points of the rn^2-dimensional space V over F defined by sets A_i $(i=1, \cdots, r\geqq 2)$ which fail to generate F_n form a proper algebraic subvariety of V. (3) We have $P_n\in M_n$, while $P_n(a, b)\neq 0$ for conjugate elements a, $b\in F_{n+1}$. (4) Every finite-dimensional central division algebra over F is generated by two conjugate elements. The paper concludes with further specific examples of elements of M_n. *M. F. Smiley* (Iowa City, Ia.).

Dias Agudo, F. R. The groups with operators and the theory of matrices. Univ. Lisboa. Revista Fac. Ci. A. (2) **4** (1955), 225–244.

A linear transformation A in a vector space R can be represented as a matrix, or explicitly as an operator acting in R. The terminologies of the corresponding theories are developed in extenso in parallel columns.

J. L. Brenner (Pullman, Wash.).

Vivier, Marcel. Note sur les structures unitaires et para-unitaires. C. R. Acad. Sci. Paris **240** (1955), 1039–1041.

The paper completes two previous notes [same C. R. Acad. Sci. Paris **234** (1952), 2327–2329; **238** (1954), 1957–1959; MR **14**, 236; **15**, 847]. The relation of "exterior equivalence" is used to relate, under certain conditions, a matrix of the group of the hermitian form $I_k\oplus(-I_{n-k})$ to a unitary matrix. Results are obtained on reduced forms of matrices equivalent under left and right multiplication by unitary matrices (metrical equivalence) and on factorization of a unitary matrix. *W. Givens.*

Schutte, H. J. A theorem on the divisibility of polynomials. Tydskr. Wetensk. Kuns (N.R.) **14** (1954), 106–110. (Afrikaans)

L'A. établit que si $p(x)$ est le polynôme semi-premier (="quadratfrei") associé au polynôme f à coefficients rationnels, il existe un polynôme $h(x)$ tel que $p(h(x))$ soit divisible par f. Il s'agit d'un lemme connu (ainsi que le rappelle l'A.) lorsque p est premier [A. A. Albert, Modern higher algebra, Univ. of Chicago Press, 1937, Ch. X, § 12].

G. Papy (Bruxelles).

Schutte, H. J. Polynomials for which the Galois group is imprimitive. Tydskr. Wetensk. Kuns (N.R.) **13** (1953), 205–210. (Afrikaans)

L'A. montre que le groupe de Galois d'un polynome $f(x)$, irréductible et séparable sur un corps K, est imprimitif si et seulement s'il existe une extension normale Ω de K, dans laquelle f se décompose en un produit de facteurs irréductibles ayant tous un même degré plus grand que 1, les racines de chacun de ces facteurs constituant les diverses classes d'imprimitivité. Si Ω est l'extension normale obtenue par adjonction des coefficients des facteurs premiers au corps K, le groupe de Galois de f relativement à Ω laisse invariante chacune des classes d'imprimitivité. Il en résulte que si le groupe de Galois est imprimitif, il contient au moins un élément distinct de l'identité et qui laisse invariante chacune des classes d'imprimitivité. *G. Papy* (Bruxelles).

Sardi, Umberto. Equazioni di secondo e terzo grado nell'algebra di Study. Radici ennesime di un numero di Study. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21** (1954), 166–173.

A theory of second and third degree equations and of nth roots, is developed for the algebra of Study, a non-commutative algebra of the third order over the complex numbers. *G. B. Huff* (Athens, Ga.).

Fadini, Angelo. La risoluzione delle equazioni di secondo e terzo grado nell'algebra dei numeri triduali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21** (1954), 114–126.

The theory of equations of degree two and three is

worked out in detail for the algebra of tri-dual numbers (over the complex numbers). *G. B. Huff.*

See also: van der Waerden, p. 338; Epstein, p. 343; Chevalley, p. 345; Wall, p. 347; Taussky, p. 347; Fekete and Szegö, p. 355; Magnus, p. 362; Ostrowski, p. 398; Benedicty, p. 410; Spampinato, p. 411; Saito and Morita, p. 415.

Lattices

Jónsson, Bjarni. Distributive sublattices of a modular lattice. Proc. Amer. Math. Soc. **6** (1955), 682–688.

The principal result is: X being a nonempty subset of a modular lattice A, then the following condition is necessary and sufficient for the sublattice of A generated by X to be distributive:

$$\Big(\sum_{i=1}^{m} x_i\Big)\prod_{j=1}^{n} y_j=\sum_{i=1}^{m}\Big(x_i\prod_{j=1}^{n} y_j\Big),$$

whenever m and n are positive integers and

$$x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in X.$$

This theorem is applied to the following special cases: (a) X has three elements. (b) X has four elements. (c) X is generated by the set $X_1\cup X_2\cup\cdots\cup X_p$, where the X_i are nonempty linearly ordered subsets of A. (d) X is generated by the set $B\cup C$, where B, C are distributive sublattices of A. *M. Novotný* (Brno).

Balachandran, V. K. On complete lattices and a problem of Birkhoff and Frink. Proc. Amer. Math. Soc. **9** (1955), 548–553.

An element a of a lattice L is called join irreducible if $a_1\cup a_2=a$ implies a_1 or $a_2=a$, and join prime if $a_1\cup a_2\geqq a$ implies $a_1\geqq a$ or $a_2\geqq a$. Similarly, a is called completely join irreducible if (for all existing joins $\bigcup_i a_i$) $\bigcup_i a_i=a$ implies some $a_i=a$, and completely join prime if $\bigcup_i a_i\geqq a$ implies some $a_i\geqq a$. The concepts "meet irreducible", "meet prime", "completely meet irreducible" and "completely meet prime" are defined dually. The meet $\bigcap_i a_i$ is said to be distributive if for arbitrary element a there holds the equality: $a\cup(\bigcap_i a_i)=\bigcap_i(a\cup a_i)$. A lattice will be said to be infinitely meet distributive if all (existing) meets $\bigcap_i a_i$ in it are distributive.

A characterization of an infinitely meet distributive lattice satisfying the ascending chain condition and having the zero element is given. The following assertions concerning a complete lattice L are all equivalent: (i) Every element of L is a join of a finite number of join primes. (ii) L is distributive and every element of L is a join of a finite number of join irreducible elements. (iii) The lattice L^* of ideals of L is infinitely meet distributive. (iv) Every completely meet irreducible ideal of L is a completely meet prime ideal. The complete, complemented lattices satisfying the condition (iv) are precisely the finite Boolean algebras. This is a correction of an erroneous assertion of Birkhoff and Frink [Trans. Amer. Math. Soc. **64** (1948), 299–316; MR **10**, 279]. *M. Novotný* (Brno).

Benado, Mihail. Sur un problème de M. Garrett Birkhoff. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 703–739. (Romanian. Russian and French summaries)

Using the concept of multilattice introduced by himself elsewhere [same Bul. **5** (1953), 41–48; MR **16**, 668], and defining suitably "principal quasi-linear valuations", the author discusses extensions to $n=3$ variables of the connection pointed out by the reviewer [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, Ch. 9, 13; MR **10**, 673] between solutions of the wave equation $u_{tt}=u_{xx}$, relativistic time, characteristics, and valuations in the sense of lattice theory. *G. Birkhoff* (Cambridge, Mass.).

Ellis, David. IPIC representation of lattice automorphisms. Publ. Math. Debrecen **3** (1954), 217–220 (1955).

Let $A(L)$ denote the automorphism group of a lattice L with O and I, and let $L\times L^*$ be the cardinal product of L and its dual L^*. For any $\alpha\in A(L)$ the mapping defined by $f_\alpha(x, y)=0$ if $\alpha(x)\leqq y$, and $f_\alpha(x, y)=1$ otherwise, is isotone from $L\times L^*$ onto the 2-element lattice 2. The author characterizes abstractly the class of all such "IPIC" mappings, and the group multiplication $f_\alpha f_\beta=f_{\alpha\beta}$. *G. Birkhoff* (Cambridge, Mass.).

Durst, L. K. On certain subsets of finite boolean algebras. Proc. Amer. Math. Soc. **6** (1955), 695–697.

Represent each element u of B_n, a boolean algebra of dimension n, by an n-digit binary number. Let G_n be the group of transformations, $B_n\to B_n$, which permute the components of u or interchange 0,1 in certain components of every u in B_n. The order of G_n is $n!2^n$. Two subsets of B_n are congruent modulo G_n if one is the image of the other under a transformation by an element of G_n. The order of a subset of B_n is the number of elements in the subset. $N_n^{(s)}$ is the number of congruence classes of subsets of order s. Theorem:

$$N_n^{(3)}=\sum_{k=2}^{n}\left\{\left[\frac{k}{3}\right]+\sum_{r=0}^{[k/3]}\left[\frac{k-3r}{2}\right]\right\}$$

S. Sherman (Philadelphia, Pa.).

See also: Jaffard, p. 346; Lesieur, p. 347; Nakano, p. 387; Andreoli, p. 398; Marinescu, p. 421; Hohn and Schissler, p. 436; Samson, and Mills p. 436.

Rings, Fields, Algebras

Nakayama, Tadasi. Über die Kommutativität gewisser Ringe. Abh. Math. Sem. Univ. Hamburg **20** (1955), 20–27.

The author generalizes a result of the reviewer's to: let R be a ring with center Z and suppose that for every simply generated subring $K=Z[c]$ there exist a finite number of elements $\alpha_1(K), \cdots, \alpha_r(K)$ in Z such that for every a in K there exists an element $p(a, K)$ in the subring generated by $a, \alpha_1(K), \cdots, \alpha_r(K)$ so that $a^2p(a, K)-a\in Z$; then R is commutative. The proof follows the reviewer's proof closely; however, in the disposing of the division-ring case the author proves a generalization of a valuation-theory result due to Nagata-Nakayama-Suzuku; a different argument than previously used is needed in the final reduction to the case $a^{m(a)}-a\in Z$. *I. N. Herstein.*

Kantz, Georg. Über den Typus eines Zerlegungsringes. Monatsh. Math. **59** (1955), 104–110.

Let J be an integral domain with unit element. An element π in J is defined to be a prime element if when-

ever $\pi|\alpha\beta$ and $\pi\nmid\alpha$ then $\pi|\beta$. An H-ring is a principal ideal ring, and a Z-ring one in which every element has a unique-factorization into prime elements (every H-ring is a Z-ring but not conversely). The author proves: 1. If a Z-ring is an algebraic extension of an H-ring then it is an H-ring. From this the author deduces: 2. A Z-ring is either an H-ring or a simple or multiple transcendental extension of an H-ring. He then characterizes those integral domains with unit element which are Z-rings precisely as those for which a rational-integer-valued function χ can be defined so that 1) $\chi(\alpha)>0$ for $\alpha\neq 0$, $\chi(0)=0$; 2) $\chi(\alpha\beta)=\chi(\alpha)\chi(\beta)$; 3) $\chi(\eta)=1$ if and only if η is a unit; and 4) if α, β are neither divisible by the other, then there exist two elements μ, ν and a ϱ prime to α so that $\mu\alpha+\nu\beta=\varrho\gamma$ and $0<\chi(\gamma)<\min[\chi(\alpha),\chi(\beta)]$. *I. N. Herstein.*

Kantz, Georg. Über Integritätsbereiche mit eindeutiger Primelementzerlegung. Arch. Math. 6 (1955), 397–402.

In the paper reviewed above the author has characterized those integral domains with unit in which a unique factorization into primes exists. The author re-proves this theorem. *I. N. Herstein* (Philadelphia, Pa.).

Rédei, Ladislaus. Die Holomorphentheorie für Gruppen und Ringe. Acta Math. Acad. Sci. Hungar. 5 (1954), 169–195. (Russian summary)

The author is concerned with what would be the appropriate analogue in ring theory for certain concepts used in the theory of groups. His viewpoint is to describe the group-theoretic property in terms of the behavior in the Schreier extensions of the group and in its holomorph and then study what ring-theoretic properties behave this way in the Schreier extensions of the rings and in its holomorphs. To do this the author defines the holomorphs of a ring and proceeds with the program outlined above. For instance, a complete group (i.e., without center and without outer automorphisms) is a direct factor of every one of its Schreier extensions. Thus the ring-theoretic analogue of complete group becomes a ring with unit element. The analogue of characteristic subgroup becomes an ideal which stays an ideal in every Schreier extension (actually it is enough that this happens in every holomorph); a group without center goes over to a ring with no absolute zero divisors. The author proves, using his method, a result of Nagata that a prime ideal is characteristic (in the above sense). He also shows that all the holomorphs of an integral domain are commutative, and characterizes when a zero-ring has only one holomorph. *I. N. Herstein* (Philadelphia, Pa.).

Rédei, László. The theory of holomorphs of groups and rings. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3 (1954), 25–48. (Hungarian)

Hungarian version of the paper reviewed above.

Szendrei, J. Eine neue Definition des Holomorphes der Gruppe und der Holomorphe des Ringes. Acta Math. Acad. Sci. Hungar. 5 (1954), 197–201. (Russian summary)

The author introduces a definition for the holomorphs of a ring (more than one holomorph may exist) and shows it to be equivalent to the definition given by Redei in the two papers reviewed above. *I. N. Herstein.*

Szendrei, János. A new definition of group holomorph and ring holomorphs. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4 (1954), 237–240. (Hungarian)

Hungarian version of the paper reviewed above.

Kertész, A. Modules and semi-simple rings. I. Publ. Math. Debrecen 3 (1954), 289–296 (1955).

The author is concerned with finding various characterizations of semi-simple rings satisfying the descending chain condition on right ideals. A sample of one such characterization is, for instance, the following: A ring R with unit element and descending chain condition is semi-simple if and only if it satisfies any of the following conditions: 1) R is a direct sum of its minimal left ideals; 2) the left annhilator of any non-zero element of R is the intersection of a finite number of maximal left ideals of R; 3) every maximal independent system over R is a basis of the additive group of R (over R); 4) for any left ideal L of R there exists a left ideal K of R for which $R=L\oplus K$.

I. N. Herstein (Philadelphia, Pa.).

Kokoris, Louis A. Power-associative rings of characteristic two. Proc. Amer. Math. Soc. 6 (1955), 705–710.

Theorem: Let A be a commutative ring of characteristic 2. If $(x^2y)y=(y^2x)x$ and $x^{2^{n-1}}x^{2^{n-1}}=x^{2^n}$ for all x, y in A and all positive integers n, then A is power-associative. Corollary: Let F be a field of characteristic 2 containing at least 4 elements. Then a commutative algebra A over F is power-associative in case $x^{2^{n-1}}x^{2^{n-1}}=x^{2^n}$ for all x in A and all n. An example shows that the assumption on 2^nth powers is necessary. *R. D. Schafer* (Storrs, Conn.).

Kokoris, Louis A. On u-stable commutative power-associative algebras. Proc. Amer. Math. Soc. 6 (1955), 702–704.

Let A be a u-stable simple commutative power-associative algebra of degree 2 over its center F of characteristic 0. It is proved in this paper that A is a Jordan algebra. [Reviewer's note: The author has subsequently announced that the hypothesis of u-stability can be removed.] *R. D. Schafer* (Storrs, Conn.).

Nagata, Masayoshi. Corrections to my paper "On Krull's conjecture concerning valuation rings." Nagoya Math. J. 9 (1955), 209–212.

See same J. 4 (1952), 29–33; MR 13, 904.

★ **Chevalley, Claude. The construction and study of certain important algebras.** The Mathematical Society of Japan, Tokyo, 1955. vi+64 pp. $1.50.

This little book reproduces lectures given by Chevalley at the University of Tokyo in 1954 on graded algebras, tensor algebras, and Clifford algebras. It provides, in elegant form and great generality, a lucid and self-contained introduction to these topics. There are four chapters: I. Graded algebras, II. Tensor algebras, III. Clifford algebras, IV. Some applications of exterior algebras.

In Chapter I free algebras and universal algebras over a ring A with 1 are defined, and the existence and uniqueness of free algebras proved. A Γ-graded algebra E, for Γ an arbitrary additive group, is defined to be an algebra over A which has a direct sum decomposition $E=\sum_{\gamma\in\Gamma}E_\gamma$ as a (unitary) module, where the E_γ's are submodules of E satisfying $E_\gamma E_{\gamma'}\subseteq E_{\gamma+\gamma'}$. As particular instances of Γ-graded algebras, one has graded algebras and semi-graded algebras. Let Γ_+ be a subgroup of Γ of index 2. Then there is an associated semi-gradation of E and an involutory automorphism (the main involution J) of E. For any homomorphism φ of E into a Γ-graded algebra E', a φ-derivation of E into E' is a linear mapping, homogeneous of some degree ν (that is, $D(E_\gamma)\subseteq E'_{\gamma+\nu}$ for all $\gamma\in\Gamma$), such that

$D(xy)=D(x)\varphi(y)+\varphi(J^{\nu}x)D(y)$ for all $x, y \in E$, where J^{ν} is the identity or J according as ν is or is not in Γ_+. The notion of φ-derivation is fundamental for the remainder of the book.

In chapters II–IV, A is assumed to be commutative. For any module M over A, the tensor algebra T over M is defined, and its existence and uniqueness proved. Its structure as a graded algebra $T=\sum_\lambda T_\lambda$ is exhibited, and it is shown that, if λ is a linear mapping of M $(=T_1)$ into $T_{\nu+1}$ (ν any integer ≥ -1), then λ may be extended uniquely to a derivation of degree ν in T. The tensor product $M\otimes N$ of two modules M, N over A is defined, and for semi-graded algebras E, E' over A, the tensor product $E\otimes E'$ of the modules E, E' is given a structure of semi-graded algebra.

Let $f(x)$ be a quadratic form on a module M over A. The Clifford algebra of f is defined as $C=T/I$, where T is the tensor algebra over M and I is the ideal of T generated by all elements of the form $x\otimes x-f(x)\cdot 1$, $x \in M$. The exterior algebra E over M is the Clifford algebra of $f=0$. C is a semi-graded algebra, and E a graded algebra. If π is the natural homomorphism of T onto C, the kernel of π in M is not always 0, so that in general M and $\pi(M)$ cannot be identified and M imbedded in C. However, it is proved that this can be done in the following important cases: (1) the exterior algebra E over M; (2) M has a base consisting of a finite number of mutually orthogonal elements; or (3) A is a field. If M is imbedded in C, then there is an anti-automorphism on C of order 2 (the canonical-automorphism) leaving the elements of M fixed. Chapter III concludes with a quick sketch of orthogonal groups and spinors. Chapter IV treats applications of exterior algebras to Plücker coordinates, the exponential mapping, determinants, and algebraic topology. *R. D. Schafer* (Storrs, Conn.).

Epstein, Marvin P. On the theory of Picard-Vessiot extensions. Ann. of Math. (2) **62** (1955), 528–547.

L'A. appelle extension de Picard-Vessiot (PV-ext.) d'un corps différentiel F (de caractéristique nulle) tout surcorps (différentiel) G de F obtenu par adjonction à F d'un système fondamental de solutions d'une équation différentielle ordinaire linéaire et homogène sur F, et dont le corps des constantes D soit une extension algébrique normale du corps des constantes C de F. La théorie classique de Kolchin [Ann. of Math. (2) **49** (1948), 1–42; Amer. J. Math. **75** (1953), 753–824; MR **9**, 561; **15**, 394, 1140] suppose $C=\bar{C}$ (la barre désignant la fermeture algébrique), ce qui entraîne $C=D$. Le but du présent travail est de se libérer de cette restriction. La PV-ext. $G(\bar{D})$ de $F(\bar{D})$ relève de la théorie de Kolchin tandis que la PV-ext. G de $F(D)$ se trouve dans une situation intermédiaire $(C=D)$. L'A. édifie sa théorie en utilisant des résultats de Kolchin $(C=\bar{C})$ et des résultats concernant le cas intermédiaire $(C=D)$ qu'il établit au préalable.

Le groupe $\mathcal{G}$ des automorphismes de G/F est isomorphe à un groupe algébrique de matrices sur C que nous identifierons ici à $\mathcal{G}$. L'A. indique les relations entre ce groupe et les groupes analogues $\mathcal{G}'$ et $\mathcal{G}''$ concernant $G/F(D)$ et $G(\bar{D})/F(\bar{D})$. En particulier, les matrices de $\mathcal{G}''$ sont celles de $\mathcal{G}''$ dont les coefficients appartiennent à D. Appelant corps fixe de G/F tout corps intermédiaire H qui soit l'ensemble des invariants pour le groupe des automorphismes de G/H, il existe une correspondance biunivoque entre l'ensemble **F** des corps fixes et l'ensemble **G** des groupes correspondants. **F** est l'ensemble des corps intermédiaires qui sont fermés pour une topologie sur G définie par $\mathcal{G}$. Cette topologie permet de formuler un analogue topologique au théorème classique des extensions normales. L'ensemble **G'** des sous-groupes de $\mathcal{G}$ correspondant aux corps fixes de G/F qui contiennent D (et dont l'ensemble est désigné par **F'**) est l'ensemble des sous-groupes algébriques (sur D) de $\mathcal{G}'$.

Un corps intermédiaire de $G(\bar{D})/F(\bar{D})$ est dit dérivé si son groupe (sous-groupe algébrique sur $\bar{D}$ de $\mathcal{G}''$) est engendré par un sous-groupe algébrique (sur D) de $\mathcal{G}'$, tandis qu'un corps de G est dit dérivé s'il est la restriction à G d'un dérivé de $G(\bar{D})/F(\bar{D})$. Toute intersection de dérivés est un dérivé, et **F'** est l'ensemble des dérivés de G. Finalement, **G** est l'ensemble des sous-groupes de $\mathcal{G}$ dont la restriction à $\mathcal{G}'$ soit un sous-group algébrique de $\mathcal{G}'$, et **F** est l'ensemble des corps intermédiaires H de G/F qui engendrent un dérivé H' qui soit une extension normale de H telle que tout automorphisme de H'/H soit extensible à G. *G. Papy* (Bruxelles).

Jaeger, Arno. On partial differential equations in a field of prime characteristic. Canad. J. Math. **7** (1955), 539–542.

Let F be an extension of a field K of characteristic $p>0$ with separating transcendence base $x_1, \cdots, x_n$; let D_i be the derivation of F/K such that $D_i x_j=\delta_{ij}$. The author shows that the derivation $d=\sum_{1\leq i\leq n} x_i^{p+1}D_i$ has the property that $d, d^p, \cdots, d^{p^{n-1}}$ is a base of the vector space over F of all derivation of F/K. *E. R. Kolchin.*

See also: van der Waerden, p. 338; Taussky, p. 347; Uchiyama, p. 349; Ghika, p. 385; Pickert, p. 399; Bruck, p. 400; Hall, p. 400; Wesson, p. 400; Ryser, p. 401; Benedicty, p. 410.

Groups, Generalized Groups

Rédei, Ladislaus. Neuer Beweis des Hajósschen Satzes über die endlichen Abelschen Gruppen. Acta Math. Acad. Sci. Hungar. **6** (1955), 27–40. (Russian summary)

In an earlier paper [Monatsh. Math. **53** (1949), 221–226; MR **11**, 319] the author (i) proved the theorem of Hajós for p-groups and (ii) showed how the general case could be deduced from this. He now gives simpler proofs of both (i) and (ii) in which the group algebra is introduced only at the final stage of (i); a final section includes a somewhat complicated argument due to Szele showing how this can be avoided. The actual theorem proved is a sharper form of Hajós's theorem. If the abelian group G of order $e_1e_2\cdots e_n$ can be expressed as a direct product of simplexes

$$(*) \qquad G=(\alpha_1)_{e_1}(\alpha_2)_{e_2}\cdots(\alpha_n)_{e_n}$$

this is called a Hajós representation of G. A set of n elements α_i and exponents e_i $(i=1, 2, \cdots, n; e_i\geq 2)$ is called an ordered normal system for the group $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ generated by the α_i, when the order of $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}=e_1e_2\cdots e_k$ for $k=1, 2, \cdots, n$. The chain of subgroups

$$\varepsilon\subset\{\alpha_1\}\subset\{\alpha_1, \alpha_2\}\subset\cdots\subset\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

then corresponds to the sequence of relative indices $e_1, e_2, \cdots, e_n$. The theorem proved is that (*) is a Hajós representation if and only if, by some permutation of the suffixes, the set of elements α_i and their associated exponents e_i form an ordered normal system. This result includes Hajós's theorem. *R. A. Rankin* (Glasgow).

Rédei, Ladislaus. Zetafunktionen in der Algebra. Acta Math. Acad. Sci. Hungar. **6** (1955), 5–25. (Russian summary)

Let n be a non-negative integer and $\mathfrak{N}$ the set of numbers $1, 2, \cdots, n$. Let $A_1, A_2, \cdots, A_n$ be subgroups of a group G and, for any $\mathfrak{M}\subseteq\mathfrak{N}$, let $A_{\mathfrak{M}}$ denote the subgroup generated by the elements of A_i for $i\in\mathfrak{M}$. Define

$$\varrho(z)=\varrho(z; A_1, A_2, \cdots, A_n)=\sum_{\mathfrak{M}\subseteq\mathfrak{N}}(-1)^{(\mathfrak{M})}(A_{\mathfrak{M}})^{-z},$$

where (S) denotes the number of elements in a set S and $\mathrm{Re}\, z>0$. The zeta-function associated with the subgroups A_i is $\zeta(z)=\{\varrho(z)\}^{-1}$. These definitions apply also to rings and other algebraic structures, and limiting processes can be introduced as $n\to\infty$. The ordinary Riemann zeta-function is then a special case.

The author's main results apply to the case when G is a finite abelian group. To begin with, seven fairly straightforward theorems are proved, of which the following (Satz 5) is an example:

$$\sum_{\mathfrak{M}\subseteq\mathfrak{R}}(A_{\mathfrak{M}})^{-z}\varrho(z; A_xA_{\mathfrak{M}}/A_{\mathfrak{M}}, A_yA_{\mathfrak{M}}/A_{\mathfrak{M}}, \cdots)=1,$$

where $x, y, \cdots$ run through all the elements of $\mathfrak{N}-\mathfrak{M}$.

For an abelian p-group P with t invariants, define

$$\psi(z, P)=\prod_{i=0}^{t-1}(1-p^{i-z}),$$

and if G is the direct product of s groups $P_1, P_2, \cdots, P_s$, where P_k is a p_k-group and the p_k are different primes, put

$$\psi(z, G)=\prod_{k=1}^{s}\psi(z, P_k).$$

The author's main theorem (Satz 8) states that

$$\varrho(z; A_1, A_2, \cdots, A_n)=\sum(H)^{-z}\psi(z, G/H)\quad(n\geqq 1),$$

where the summation is over those subgroups H of G which contain no A_i. This is proved with the aid of Delsarte's generalization of the Möbius function to abelian groups [Ann. of Math. (2) **49** (1948), 600–609; MR **10**, 9]. It follows fairly easily from this and Satz 5 that

$$0\leqq\varrho(z; A_1, \cdots, A_n)<1,$$

for $n\geqq 1$, $z=1, 2, 3, \cdots$. This result the author calls the inertia theorem for finite abelian groups. When $n\leqq 2$ it holds also for all real $z\geqq 1$, but does not hold in this general form for all n. In fact, no positive constants c, C exist such that $-c<\varrho(z)<C$, for all real $z\geqq 1$.

Various examples and further results are given. In particular it is mentioned that the results can be established without appealing to the basis theorem for finite abelian groups. Misprints occur on pp. 11 (lines 18, 21), 17 (lines 2, 8, 22, 24), 19 (line 2). *R. A. Rankin.*

Hall, P., and Higman, Graham. On the p-length of p-soluble groups and reduction theorems for Burnside's problem. Proc. London Math. Soc. (3) **6** (1956), 1–42.

The reviewer can do no more than just indicate the wealth of material contained in this investigation. A [finite] group G is termed p-soluble (following Čunihin) if every composition factor of G is either a p-group or of order prime to the prime p. Such a p-soluble group G possesses the characteristic series

$$1=P_0\leqq N_0<\cdots<P_i<N_i<P_{i+1}<\cdots<P_h\leqq N_h=G,$$

where N_i/P_i is the product of all the normal subgroups of G/P_i whose order is prime to p and where P_{i+1}/N_i is the product of all the normal p-subgroups of G/N_i. The main result of this investigation relates this invariant h of the p-soluble group G to the invariants of the p-Sylow subgroups of G in the following way: If p^b is the order of the p-Sylow subgroups of G, then $\frac{1}{2}h(h+1)\leqq b$; and if c is the class of the p-Sylow subgroups of G [=length of descending central chain], then $h\leqq c$. If p is odd, and if d is the length of the derived chain of the p-Sylow subgroups of G, then $h\leqq d$. If p^e is the maximum order of the elements in the p-Sylow subgroups of G, then $h\leqq e$, if p is odd, but not of the form 2^j+1; and $\frac{1}{2}(h+1)\leqq e$, if $p=2^j+1$. The basis for the proofs of these theorems is the following remark: If F/N_0 is the Frattini subgroup of P_1/N_0 [=intersection of maximal subgroups of P_1/N_0], then G/P_1 is essentially the same as a group of automorphisms of the elementary abelian p-group P_1/F, since the latter group is its own centralizer in G/F. This remark reduces in a way the whole problem to a study of p-soluble linear groups. Let us remark furthermore that most of the above inequalities are best possible. If G is a soluble group, then G is p-soluble for every prime p; and an invariant $h(p)$ may be defined for every prime divisor p of the order of G. If $p_0, p_1, \cdots, p_r$ are all the prime divisors of the order of the order of the soluble group G, then the authors prove the following inequality:

$$h(p_0)\leqq\prod_{i=1}^{r}[h(p_i)+1].$$

If furthermore $c(2)$ is the length of the descending central chain of the 2-Sylow subgroups and $d(p)$ the length of the derived series of the p-Sylow subgroups of the soluble group G, then the length d of the derived series of G satisfies

$$d\leqq c(2)+\sum_{2<p}\tfrac{1}{2}d(p)[d(p)+1].$$

The power of these results may be gauged from the following application which seems to constitute the first real progress towards a solution of Burnside's celebrated conjecture. Denote by (S. e) the proposition: to every positive integer k there exists a finite soluble group K of exponent e [$K^e=1$] and generated by k elements such that every finite soluble group of exponent e and generated by k elements is a homomorphic image of K. Then the authors prove the following reduction theorem: (S. e) is true, if (S. q) is true for every prime power divisor q of e. This permits, for instance, proof of the validity of Burnside's conjecture for exponent 6. *R. Baer.*

Hall, P. Finiteness conditions for soluble groups. Proc. London Math. Soc. (3) **4** (1954), 419–436.

The principal finiteness conditions that are studied in this paper are the following: FG (the group is finitely generated), Max-n (the maximal condition holds for normal subgroups), Max (the maximal condition holds for all subgroups), Max-r (the maximal condition holds for the right ideals of the group ring). For soluble groups FG, Max-n, and Max are in strictly decreasing order of generality. But the author proves by arguments that resemble the ideas of Hilbert's Basis Theorem that for soluble groups Max and Max-r are equivalent. Some recent results on groups of this class which have previously been studied by the reviewer are contained in papers by Mal'cev [Mat. Sb. N.S. **28(70)** (1951), 567–588; MR **13**, 203], Smirnov [ibid. **32(74)** (1953), 365–384; MR **14**, 947], and Baer [Math. Ann. **129** (1955), 139–173; MR **16**, 994]. The author uses the name poly-cyclic for groups of this class, because they arise from the unit subgroup by a

finite number of cyclic extensions. He also proves that a finitely generated extension of an abelian group by a poly-cyclic group satisfies Max-n. This result illustrates the main problem of the paper: What varieties of soluble groups satisfy Max-n? (A variety of groups is an equationally defined class of groups.) To anticipate the main result, consider the class of groups that satisfy the condition $G''=1$, in other words, the metabelian groups. There are only a countable number of non-isomorphic finitely generated metabelian groups, and each of them satisfies Max-n. In contrast, consider the class of groups defined by $[G'', G]=1$, that is, the groups for which the factor-group of the centre is metabelian. Here the number of distinct types of finitely generated groups is uncountable. Indeed, the author shows much more: an arbitrary non-trivial countable abelian group can be chosen as the centre, and for every such choice the number of distinct two-generator groups in the variety $[G'', G]=1$ is uncountable. The two conditions $G''=1$ and $[G'', G]=1$ are special cases of a much more general set-up.

Let $G=\gamma_1(G)\geqq\gamma_2(G)\geqq\cdots$ be the lower central series of G. If φ and ψ are two subgroup functions, assigning to an arbitrary group G subgroups $\varphi(G)$ and $\psi(G)$, respectively, then $[\varphi, \psi]$ shall denote that subgroup function which assigns to G its subgroup $[\varphi(G), \psi(G)]$ generated by all commutators $[x, y]$ with $x \in \varphi(G)$, $y \in \psi(G)$. Subgroup functions can be partially ordered by defining $\varphi\leqq\psi$ if $\varphi(G)\subseteq\psi(G)$ for every group G. The class C of "commutator subgroup functions" is defined to contain precisely the following subgroup functions: γ_1 (the identity function), and with any two functions φ and ψ also $[\varphi, \psi]$. For example, $\gamma_{k+1}=[\gamma_k, \gamma_1]$ for every k, and also $\delta_k=[\gamma_k, \gamma_k]$. Clearly $\delta_1(g)=G'$ and $\delta_2(G)=G''$. Now it turns out (by an induction argument) that all the functions φ in C fall into two disjoint kinds: (i) $\varphi\geqq\delta_k$ for some k; (ii) $\varphi\leqq[\delta_2, \gamma_1]$. If φ belongs to the first kind, then among the groups of the class $\varphi(G)=1$ there is only a countable number satisfying FG, and then FG implies Max-n. But if φ belongs to the second kind, then the number of FG groups in the class $\varphi(G)=1$ is not countable, and FG does not imply Max-n. In particular, if F_2 is the free group on two generators, $F^3/\varphi(F_2)$ does not satisfy Max-n. The author provides lucid illustrations of his theorems and the limitations of their validity by means of wreath groups (Kranzgruppen).

K. A. Hirsch (London).

irsch, Kurt A. Über lokal-nilpotente Gruppen. Math. Z. 63 (1955), 290–294.

The author proves the product of two normal, locally nilpotent subgroups is itself locally nilpotent. It follows at once that a given group G possesses a uniquely determined maximal, locally nilpotent normal subgroup in which all locally nilpotent, normal subgroups of G are included. He shows that the normalizer tower of a maximal, locally nilpotent subgroup breaks off in one step. These results combine readily to give a simple proof of a result of Plotkin [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 639–641; MR **12**, 587] that a group G in which every proper subgroup does not coincide with its normalizer is locally nilpotent. He completes earlier work [Math Nachr. 4 (1951), 47–49; MR **12**, 587] by showing that every finitely generated nilpotent group satisfies the maximal condition for subgroups. *F. Haimo* (St. Louis, Mo.).

Mal'cev, A. I. Two remarks on nilpotent groups. Mat. Sb. N.S. **37(79)** (1955), 567–572. (Russian)

The first remark is centered about the Theorem: In V, a free nilpotent group of class n, a subset can act as a set of free generators for a nilpotent subgroup of class n if and only if the image of the subset in the factor-commutator group of V is a linearly independent set. The second remark gives a detailed procedure for determining whether an element is the identity in such a group V with a finite number of free generators. [See R. C. Lyndon, Proc. Amer. Math. Soc. 3 (1952), 579–583 [MR **14**, 242] where the decidability of this word-problem had been established.] *F. Haimo* (St. Louis, Mo.).

Crouch, Ralph B. Monomial groups. Trans. Amer. Math. Soc. **80** (1955), 187–215.

Let B, C be cardinals, of which C at least is infinite, U a set of cardinal B, and H a group. Denote by $V(B, C)$ the subgroup of the direct product of B copies of H, indexed by U, consisting of those elements with less than C components differing from 1. If G is a group of permutations of U, then G acts naturally as a group of automorphisms of $V(B, C)$, and so the holomorph of $V(B, C)$ contains a subgroup Σ which is essentially a product $V(B, C)G$. This group is here studied in case G is either (i) $S(B, D)$, the group of all permutations of U which disturb fewer than D elements, D being a third infinite cardinal, or (ii) $A(B)$, the group of all permutations which disturb only a finite set of elements, and permute those evenly. The first case was treated by Ore [same Trans. **51** (1942), 15–64; MR **3**, 197] if B is finite and G is of finite order.

The principal matters discussed are (i) conditions under which Σ splits regularly over $V=V(B, C)$, i.e. under which every complement of V is conjugate to G, (ii) the description of the normal subgroups of Σ, and (iii) conditions under which V is a characteristic subgroup of Σ. *Graham Higman* (Oxford).

Chevalley, Claude. Invariants of finite groups generated by reflections. Amer. J. Math. **77** (1955), 778–782.

Soit G un group fini de transformations linéaires d'un espace vectoriel V sur un corps K de caractéristique 0. On suppose G engendré par des réflexions (involutions dont les éléments invariants forment un hyperplan). Présant le théorème classique des invariants pour les groupes finis, l'auteur montre que l'algèbre J sur K des invariants de G est engendré par l'unité et par un nombre d'invariants homogènes, algébriquement indépendants, égal à la dimension n de V. En outre, si $I_1, \cdots, I_n$ est un tel système de générateurs, m_i $(1\leqq i\leqq n)$ le degré de I_i, le produit $\prod_{i=1}^{n}(1-t^{m_i})$ est égal au produit de $(1-t)^n$ par un polynôme où le coefficient de t^k est la dimension des éléments homogènes de degré k dans l'algèbre S/F, S étant l'algèbre des polynômes sur V et F l'idéal homogène de S engendré par les invariants (polynômes) de G. Il en résulte que le produit $m_1m_2\cdots m_n$ est égal à l'ordre du groupe G; enfin l'auteur prouve que la représentation linéaire naturelle de G dans l'espace vectoriel S/F est équivalente à la représentation régulière de G.

J. Dieudonné (Evanston, Ill.).

Green, J. A. The characters of the finite general linear groups. Trans. Amer. Math. Soc. **80** (1955), 402–447.

In this paper the irreducible complex characters of the group $G_n=\mathrm{GL}(n, q)$ of all nonsingular matrices of degree n with coefficients in the finite field $\mathrm{GF}(q)$ of q elements are determined. Three main steps are involved in the construction. First the author provides a source of characters by proving the following result: For each x in

G_k, let $\xi_1(x), \xi_2(x), \cdots, \xi_k(x)$ be the latent roots of x. Let $S(t_1, t_2, \cdots, t_k)$ be any symmetric polynomial in $t_1, t_2, \cdots, t_k$ with rational integer coefficients. Then, if $\theta: \alpha \to \alpha_0$ is a homomorphism of the multiplicative group of $GF(q^{k!})$ into the group of complex roots of 1, the function χ, defined by $\chi(x)=S(\xi_1(x)_\theta, \cdots, \xi_k(x)_\theta)$, is a character of G_k. The characters so obtained are then used as building blocks to yield other characters using the Frobenius method [S.-B. Preuss. Akad. Wiss. **1898**, 501–515] of induced characters applied to suitably chosen subgroups of G_n. A typical such subgroup is obtained by taking a partition $n=s_1+s_2+\cdots+s_j$ of n and then considering those matrices of G_n which have blocks of size $s_k \times s_k$ ($k=1, 2, \cdots, j$) down the main diagonal with zeros below it. The third step involves the decomposition of the characters thus obtained into their irreducible parts. Here the techniques used are similar to those of Frobenius [ibid. **1900**, 516–534] in his work on the symmetric groups. More specific details of the construction cannot be given here. An interesting feature of the final result is a so-called degeneracy rule, whereby the values of an irreducible character at all elements of G_n can be computed from its values at the principal elements: those with n distinct latent roots. Involved in this rule are certain polynomials Q_ϱ^λ defined for all partitions λ and ϱ of k; $k=1, 2, \cdots, n$. A table of these Q_ϱ^λ for $1 \leq k \leq 5$ is included. [Misprint: on p. 409, l. 27, replace Σ by $\prod$.]

R. Steinberg (Princeton, N.J.).

Jaffard, Paul. Extension des groupes réticulés et applications. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 197–222 (1955).

This paper is concerned exclusively with abelian groups, and the term abelian will be omitted. If H and Q are given partially ordered abelian groups (p.o. groups), a p.o. group G is called an ordered extension of H by Q (of Q by H in the author's terminology) if it contains an isolated (=convex) subgroup H' order-isomorphic with H, such that G/H' is order-isomorphic with Q. Here G/H' is ordered as usual by declaring a coset to be positive if it contains a positive element of G. A general solution of the extension problem, even for non-abelian p.o. groups, has been given by L. Fuchs [Acta Math. Acad. Sci. Hungar. **1** (1950), 118–124; MR **13**, 436]. As in the Schreier theory, G can be described as the set of ordered pairs (a, x) with $a \in H$, $b \in Q$, defining $(a, x)+(b, y)$ to be $(a+b+f(x,y), x+y)$, with $f(x, y)$ a factor set of $Q \times Q$ in H satisfying $f(x, 0)=f(0, y)=0$. Fuchs shows that all possible orderings of G are obtained by assigning to each pair x, y of elements of Q a subset $P_{x,y}$ of H, these sets $P_{x,y}$ being subject to five conditions, and defining $(a, x) \geqq (b, y)$ if $-b+a \in P_{x,y}$. If we set $\psi(x)=P_{x,0}$, and assume G abelian, Fuchs' five conditions reduce to three conditions on the sets $\psi(x)$, given in the present paper. The author gives a direct derivation of these without mentioning any connection between his sets $\psi(x)$ and Fuchs' sets $P_{x,y}$. For each group extension G of H by Q, the lexicographic ordering of G is defined by taking $\psi(x)=H$ for all $x>0$, and $\psi(0)=H_+$ (=set of positive elements of H). If $f(x, y)=0$, then the definition $\psi(x)=H_+$ for all $x \in Q_+$ leads to the ordered direct sum (=cardinal sum) of H and Q. Other specializations are also discussed.

A subgroup H of a p.o. group G is said to be strongly isolated if, for each $\xi \in G$, $\xi \geqq h$ for some $h \in H$ implies $\xi \in H$ or $\xi > 0$. [Note misprint in line 4, page 205: the implication goes one way only.] A strongly isolated subgroup H of G is also isolated. If G is lattice-ordered (lat.o.) then H is also a sublattice of G, hence an l-ideal in the sense of G. Birkhoff [Ann. of Math. (2) **43** (1942), 298–331; MR **4**, 3); moreover, G/H is totally ordered (t.o.) and G is a lexicographic extension of H by G/H. [Conversely, an l-ideal with these properties is strongly isolated.] In a previous paper [J. Math. Pures Appl. (9) **32** (1953), 203–280; MR **15**, 284] the author introduced the notion of a "filet" in a lat.o. group. If G is any lat.o. group, the author defines $\hat{G}$ to be the subgroup of G generated by all those elements of G_+ not belonging to a maximal filet of G [i.e. those which are not weak units]. $\hat{G}$ is a strongly isolated subgroup H of G such that $\hat{H}=H$.

Now let G be a lat.o. group having n filets (n finite and >1). If $\hat{G}=G$, then G is uniquely expressible as a cardinal sum of at least two lat.o. groups G_i such that $\hat{G}_i \neq G_i$; moreover, each G_i has less than n filets. If $\hat{G} \neq G$, then G is cardinally indecomposable, but, by the foregoing, G is a lexicographic extension of $\hat{G}$ by the t.o. group $G/\hat{G}$; in this case $\hat{G}$ has n filets, unless $\hat{G}=(0)$, which is the case if and only if G itself is t.o. Since n is finite, it follows that G can be built up in a unique fashion from a set of t.o. groups by a succession of cardinal sums and lexicographic extensions. [The author seems unaware that this result was found by Birkhoff (loc.cit.) for lat.o. groups having only a finite number of l-ideals. Jaffard's theorem, however, properly includes Birkhoff's. For if two elements of G_+ generate the same principal l-ideal, they belong to the same filet, but not conversely.] *A. H. Clifford* (New Orleans, La.).

Schieferdecker, Eberhard. Die fastperiodischen Funktionen einer Oreschen Halbgruppe. Arch. Math. **6** (1955), 428–438.

A semigroup $\mathfrak{H}$ (i.e., a set with an associative multiplication) is called an Ore semigroup if it satisfies both cancellation laws and has the property that for all $a, b \in \mathfrak{H}$ there exist $a', b' \in \mathfrak{H}$ such that $ab'=ba'$. It is well known that every Ore semigroup can be imbedded in the group $\mathfrak{Q}_r(\mathfrak{H})$ of all right quotients $a\alpha^{-1}$ ($a, \alpha \in \mathfrak{H}$) [see Ore, Ann. of Math. (2) **32** (1931), 463–477]. For an arbitrary semigroup $\mathfrak{H}$ with unit (the presence of a unit is essential), the author adopts Maak's definition of an almost periodic function. A complex function f on $\mathfrak{H}$ is almost periodic if for each $\delta > 0$ there is a finite family $\{\mathfrak{A}_j\}_{j=1}^n$ of subsets of $\mathfrak{H}$ such that (1) $\bigcup_{j=1}^n \mathfrak{A}_j = \mathfrak{H}$, (2) if $x, y \in \mathfrak{H}$ have the property that $c_0 x d_0$ and $c_0 y d_0$ lie in a single $\mathfrak{A}_j$ for some $c_0, d_0 \in \mathfrak{H}$, then $|f(cxd)-f(cyd)|<\delta$ for all $c, d \in \mathfrak{H}$. Maak has shown that this definition is equivalent to the usual one if $\mathfrak{H}$ is a group [Acta Math. **87** (1952), 33–58; MR **13**, 910]. The main result of the present paper is that if $\mathfrak{H}$ is an Ore semigroup with unit, then every almost periodic function on $\mathfrak{H}$ admits a unique almost periodic extension over $\mathfrak{Q}_r(\mathfrak{H})$. The key to the proof is a construction given by Maak (loc. cit.). For $\mathfrak{H}=[0, \infty[$ under addition and f a continuous almost periodic function on $\mathfrak{H}$, the theorem has been proved by H. Bohr [J. Reine Angew. Math. **157** (1926), 61–65]. Bohr's proof contains the essence of the present one. The author also compares $\mathfrak{Q}_r(\mathfrak{H})$ with a group constructed by Maak (loc. cit.) and shows what happens to his theorem if $\mathfrak{H}$ is not an Ore semigroup.

E. Hewitt (Princeton, N.J.).

Ivan, Ján. On the decomposition of simple semigroups into a direct product. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4** (1954), 181–202. (Slovak. Russian summary)

Let S be a finite simple semigroup without zero, in which the product of two idempotents is idempotent.

From the structure theorem of Suschkewitsch [Math. Ann. 99 (1928), 30–50] it is easy to see that $S \cong G \times E$, where G is a group and E is a semigroup of idempotents $e_{11}, \cdots, e_{mn}$ with product $e_{i\lambda}e_{j\mu} = e_{i\mu}$ $(i, j = 1, \cdots, m; \lambda, \mu = 1, \cdots, n)$. [In the reviewer's terminology, E is a finite rectangular band.] From this follows the author's main theorem: S is indecomposable into a direct product of semigroups if and only if either (1) S is an indecomposable group, or (2) the order of S is a prime. *A. H. Clifford.*

Lesieur, L. Sur les idéaux irréductibles d'un demi-groupe. Rend. Sem. Mat. Univ. Padova **24** (1955), 29–36.

Let D be a commutative semigroup with identity element, the ideals of which satisfy either the ascending or the weak descending chain condition. The paper is devoted to proving the following theorem, due to W. Gröbner [Math. Ann. **110** (1934), 197–222; Theorem 3, p. 204] for the case D a Noetherian ring. Let q be a primary ideal of D, and let p be its associated prime ideal. Then the number of components in an unshortenable representation of q as an intersection of irreducible ideals is equal to the length of the lattice of primary ideals lying between q and $q:p$. In particular (Gröbner, loc.cit., Theorem 4a), an ideal q is irreducible if and only if (1) q is primary and (2) there is no primary ideal strictly between q and $q:p$. The theorem and its proof are phrased for the (distributive) residuated lattice T_D of ideals of D. [If distributivity can be replaced by modularity, Lesieur's theorem contains Gröbner's, but this equation is not discussed.] *A. H. Clifford* (New Orleans, La.).

See also: Vivier, p. 340; Rédei, p. 342; Schieferdecker, p. 384; Libermann, p. 404; Arcidiacono, p. 437.

THEORY OF NUMBERS

★**Rusu, Eugen. Bazele teoriei numerelor. [Elements of the theory of numbers.]** Biblioteca Societătii de Ştiinţe Matematice şi Fizice din R.P.R., Bucarest, 1953. 181 pp. Lei 6.23.

This is an elementary introduction to number theory. It consists of three parts. The first is mainly an inductive exposition of fundamental concepts and uses a large number of numerical examples. It contains five chapters: Preliminary topics (divisibility, primes, residue classes); Periodicity of residues (congruences, theorems of Fermat and Wilson, residue class field mod p); The Euler function n-th power residues (primitive roots, quadratic residues, index calculus); Applications (periodic decimal fractions). The second part is a more abstract treatment of the material presented; its three chapters are: The uniqueness of decomposition into primes; The representation of primes as sum of two squares; Gaussian integers. The third part contains a (partly expository) treatment of Fermat's last theorem, of the Pell equation and of the Fibonacci numbers. Particular mention deserve the careful discussions of the Euler function, the primitive roots mod $n \neq p$ and the representation of integers as sums of two squares. The book contains a large number of exercises and tables of primes ($<4{,}000$) and smallest primitive roots, power residues, indices (mod p, $p<100$) and of smallest divisors of $n<10{,}000$. *E. Grosswald.*

Have ★**Patz, Wilhelm. Tafel der regelmässigen Kettenbrüche und ihrer vollständigen Quotienten für die Quadratwurzeln aus den natürlichen Zahlen von 1-10000.** Akademie-Verlag, Berlin, 1955. xi+1210 pp. DM 58.00.

This work may be regarded as much enlarged second edition of the author's previous volume entitled "Tafel der regelmässigen Kettenbrüche für die Quadrat wurzeln aus den natürlichen Zahlen von 1–10000 [Akademische Verlagsgesellschaft, Leipzig, 1941; MR 8, 197]." As the new title indicates, the new material included in the present table consists in the positive integers P_n and Q_n in the complete quotients $x_n = (D^{\frac{1}{2}} + P_n)/Q_n$. The former table gives only the partial quotients $[x_n]$. This new table is surprisingly larger by a factor of more than four. Of course, it is much more useful. For example, one can tell at a glance whether or not the diophantine equation $x^2 - Dy^2 = k$ $(k^2 < D)$ is possible or not. The table extends actually to 10010 and is dedicated to the memory of H. Brandt who wrote the explanatory introduction. *D. H. Lehmer* (Berkeley, Calif.).

Gloden, Albert. Table of minimal solutions of the congruence $x^4+1 \equiv 0 \pmod{p^2}$ for $4000<p<6000$. Scripta Math. **21** (1955), 218.

Wall, C. T. C. Nim-arithmetic. Eureka no. **18** (1955), 3–7.

The author sets up a one-to-one correspondence between non-negative rational integers and polynomials in GF[2, x] by means of

$$\sum a_i 2^i \rightleftarrows \sum a_i x^i \quad (a_i = 0, 1).$$

He then develops a number of known results concerning the arithmetic of GF[2, x], in particular, the explicit formula for the number of irreducible polynomials of degree n. *L. Carlitz* (Durham, N.C.).

Hanani, Haim. Enumeration of rational numbers. Riveon Lematematika **9** (1955), 23–24. (Hebrew. English summary)

Klamkin, Murray S. On some identities of Lucas. Scripta Math. **21** (1955), 213–214.

Subba Rao, K. Some summation formulae involving Fibonacci numbers. Scripta Math. **21** (1955), 214–217.

Taussky, Olga. Unimodular integral circulants. Math. Z. **63** (1955), 286–289.

The title refers to $p \times p$ integral determinants (p prime) each of whose rows is a cyclic permutation of the preceding and in which the elements (b_i) of the first row satisfy $\sum b_i = \pm 1$ and $\sum b_i \zeta^{i-1} = \varepsilon(\zeta)$, a unit in the field generated by $\zeta = \exp 2\pi i/p$. The problem considered is how to recognize those determinants for which ε is the product of a unit $\varepsilon_0(\zeta)$ and its complex conjugate $\varepsilon_0(1/\zeta)$. When $p=5$ a sufficient condition is that the determinant be symmetric and positive definite. For general p recognition involves the more difficult factorization of the matrix as the product of a circulant and its transpose. *Harvey Cohn.*

Buquet, A. Sur un critère d'indépendance de plusieurs solutions données de l'équation diophantienne en nombres rationnels $G(x) = ax^4 + bx^3 + cx^2 + dx + e = z^2$. Mathesis **64** (1955), 231–241.

Selfridge, J. L., Nicol, C. A., and Vandiver, H. S. Proof of Fermat's last theorem for all prime exponents less than 4002. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 970–973.

Two previous papers [D. H. Lehmer, E. Lehmer, and H. S. Vandiver, same Proc. **40** (1954), 25–33; Vandiver, ibid. **40** (1954), 732–735; MR **15**, 778; **16**, 13] describe calculations made on the SWAC which extend the proof of Fermat's Last Theorem to exponents up to 2521. In the present note the extension is made to 4001 using routines modified to give the greater speeds required for these larger exponents. Of the 183 prime exponents considered, 72 are irregular, thus perpetuating a probability of about $.39=1-\varepsilon^{-1}$ that a prime be irregular. All irregular primes so far discovered are properly irregular. *D. H. Lehmer* (Berkeley, Calif.).

Naor, Pinhas. A property of prime twins. Riveon Lematematika **9** (1955), 36–40. (Hebrew. English summary)

The author establishes an isomorphism between the multiplicative semigroup of natural numbers N which are relatively prime to 6, and the semigroup of integers n under the operation $n\times m=6nm+n+m$, by setting $N=|6n+1|$. Thus concepts of divisibility, primality, unique factorization etc. are inferred for the operation $\times$. In particular, N, N' are twin primes if and only if for $t=(N+N')/12$ both t and $-t$ are primes under $\times$, that is, the equation $t=|6ab+a+b|$ has no nonzero integer solutions a, b. This enables one to give a type of sieve method for twin primes from the $\times$ multiplication table. *E. G. Straus* (Princeton, N.J.).

Hampel, R. The length of the shortest period of rests of numbers n^n. Ann. Polon. Math. **1** (1955), 360–366.

Sierpiński has proved that for every natural m the integers n^n $(n=1, 2, 3, \cdots)$ form an infinite periodic sequence (mod m) [Ann. Soc. Polon. Math. **23** (1950), 252–258; MR **12**, 674]. Let S_m denote the least positive integer such that $(n+S_m)^{n+S_m}\equiv n^n \pmod m$ for all n from some point on. The author proves that if $m=\prod_{i=1}^{k} p_i^{\alpha_i}$ is the decomposition of m into prime powers, S_m is equal to the least common multiple of the numbers m, p_1-1, p_2-1, $\cdots$, p_k-1.

The first step in the proof is to show that $\phi(m)\prod_{i=1}^{k} p_i$ is divisible by S_m, $\phi(m)$ being Euler's function. Next the theorem is established for $m=p$ (this part of the argument is due to Sierpiński), then for $m=p^\alpha$, $\alpha\geqq 2$, and finally for general m. The methods of the proof lie in the theory of linear congruences; frequent use is made of the Euler-Fermat congruence and of the properties of primitive roots.

It should be pointed out that equation (7) must read: "$S_m|p^{-1}(p-1)$ for $(n, p)=1$", for the subsequent argument to be correct. *H. Halberstam* (Providence, R.I.).

Duparc, H. J. A., and Peremans, W. On theorems of Wolstenholme and Leudesdorf. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 459–465.

The authors consider the sum $T_s(M)=\sum a^{-s}$ (s an integer) extended over all $\phi(M)$ positive integers $a\leqq M$ and prime to M. The divisibility of $T_s(M)$ by powers of primes dividing M is the subject of two theorems which generalize the theorems of Wolstenholme [Quart. J. Pure Appl. Math. **5** (1862), 35–39] and Leudesdorf [Proc. London Math. Soc. **20** (1889), 199–212] and other recent results.

The first theorem is for the case $M=p^n$, p a prime. Define the function $k=k(n)$ by

$$k(n)=\begin{cases} 2n-2 & \text{if } p=2 \text{ and } s \text{ is odd,}\\ n-1 & \text{if } p-1 \text{ divides } s \text{ and } s \text{ is even,}\\ 2n & \text{if } p-1 \text{ does not divide } s+1;\ s \text{ is odd,}\\ 2n & \text{if } p|s \text{ and } s \text{ is odd,}\\ n & \text{if } p-1 \text{ does not divide } s \text{ and } s \text{ is even,}\\ 2n-1 & \text{otherwise.}\end{cases}$$

Then $T_s'(p^n)$ is divisible by p^k and in the first, second, and last cases, in definition of $k(n)$, T is not divisible by p^{k+1}. The second theorem deals with the case a general positive integer M and the divisibility of $T_s(M)$ by a prime factor of M. There is a discussion of some results of Carlitz [Amer. Math. Monthly **61** (1954), 174–176; MR **15**, 603], concerning the more general sum $\sum (a+\lambda M)^{-s}$. *D. H. Lehmer* (Berkeley, Calif.).

Dobbie, J. M. A simple proof of some partition formulae of Ramanujan's. Quart. J. Math. Oxford Ser. (2) **6** (1955), 193–196.

Bailey [same J. (2) **3** (1952), 158–160; MR **14**, 138] showed that certain well-known formulas of Ramanujan can be obtained as special cases of the identity

$$(*)\quad \sum_{-\infty}^{\infty}\left\{\frac{xq^n}{(1-xq^n)^2}-\frac{yq^n}{(1-yq^n)^2}\right\}=\frac{(x-y)(1-xy)}{(1-x^2)(1-y^2)}\prod_{1}^{\infty}\frac{(1-xyq^n)(1-x^{-1}yq^n)(1-xy^{-1}q^n)(1-x^{-1}y^{-1}q^n)(1-q^n)^4}{(1-xq^n)^2(1-x^{-1}q^n)^2(1-yq^n)^2(1-y^{-1}q^n)^2},$$

which is itself obtained from

$$\wp(u)-\wp(v)=-\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}.$$

Similarly, the reviewer [ibid. **4** (1953), 168–172; MR **15**, 201] applied the identity

$$(**)\quad 1+x^{-1}\frac{(1-x)^3}{1+x}\sum_{1}^{\infty}\frac{n^2q^{2n}}{1-q^n}(x^n-x^{-n})=\prod_{1}^{\infty}\frac{(1-x^2q^n)(1-x^{-2}q^n)(1-q^n)^6}{(1-xq^n)^4(1-x^{-1}q^n)^4}$$

which is obtained from $\wp'(u)=-\sigma(2u)/\sigma^4(u)$.

In the present paper the identities (*) and (**) are proved by a simple method (frequently employed by Jacobi and Cauchy) that does not presuppose any knowledge of elliptic functions. *L. Carlitz.*

Selberg, Sigmund. Über die Summe $\sum_{n\leqq x}\frac{\mu(n)}{n}$.

Norske Vid. Selsk. Forh., Trondheim **28** (1955), 37–41.

The author shows that the partial sums of the title cannot be of one sign for large x. *R. Bellman.*

Analytic Number Theory

Val'fiš [Walfisz], A. Z. On sums of coefficients of certain modular forms. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 417–423. (Russian)

For the entire cusp form $F(\tau)=\sum c_n \exp 2\pi i n/N$ of dimension $-k$ and order N the author gives another proof of his result [Math. Ann. **108** (1933), 75–90] $C(x)=\sum_{n\leqq x} c_n=\Omega(x^{\frac{1}{2}k-\frac{1}{4}})$. With $\tau^k F(\tau)=\sum b_n \exp[-2\pi i n/(N\tau)]$, by Riemann's transform-method he shows $\Gamma(s)\sum c_n n^{-s}=i^{-k}(N/2\pi)^{k-2s}\Gamma(k-s)\sum b_n n^{s-k}$, making applicable a general

theorem of Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1924, 137–150] which yields the main result Re $\{wC(x_n)\}>a_3 x_n^{\frac{1}{2}k-\frac{1}{2}}$, Re $\{wC(y_n)\}<-a_3 y_n^{\frac{1}{2}k-\frac{1}{2}}$ for a sequence x_n with $x_n<x_{n+1}<x_n+a_n x_n^{\frac{1}{2}}$, a like sequence y_n, positive constants a_n, and a complex constant w, depending on $F(\tau)$. [See the paper reviewed below.]

Harvey Cohn (Detroit, Mich.).

Val'fiš [Walfisz], A. Z. On sums of modules of the coefficients of certain modular forms. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 497–502. (Russian)

With the notation of the paper reviewed above, the author shows that $D(x)=\sum_{n\leq x}|c_n|\geq a_{12}x^{\frac{1}{2}k+\frac{1}{2}}$ directly from the main result quoted in the preceding review. Under the further assumption that $c_n=O(n^{\frac{1}{2}k-\frac{1}{2}+\epsilon})$, the author shows $D(x)\geq a_{13}x^{\frac{1}{2}k+\frac{1}{2}-\epsilon}$ by means of Rankin's result on $\sum|c_n|^2$ [Proc. Cambridge Philos. Soc. **35** (1939), 351–372; MR **1**, 69, 400]. Thus the author notes that the abscissa of absolute convergence of $\sum c_n n^{-s}$ is $\frac{1}{2}k+\frac{1}{2}$. The assumption is applicable to modular functions used by Eichler [Arch. Math. **5** (1954), 355–366; MR **16**, 116]; and Ramanujan's conjecture would imply $\sum_{n\leq x}|\tau(n)|\geq ax^{12/2-\epsilon}$ [see van der Blij, Math. Student **18** (1950), 83–99, p. 90; MR **13**, 328].

Harvey Cohn (Detroit, Mich.).

Tong, Kwang-Chong. On divisor problems. J. Chinese Math. Soc. **2** (1953), 258–266. (Chinese. English summary)

From the convexity of Lindelöf's function $\mu(\sigma)$ in the Riemann zeta-function [cf. Titchmarsh, The theory of the Riemann zeta-function, Oxford, 1951, Chap. V; MR **13**, 741] the author proves that

$k=$	4	5	6	7	8	9	10	11
$\alpha_k\leq$	1/2	4/7	5/8	71/107	41/59	31/43	26/35	19/25
$\beta_k\leq$	23/54	1/2	35/62	11/18	149/230			

where α_k and β_k (cf. Titchmarch, loc. cit. Chap. XII) have their usual meanings in the divisor problem.

L. K. Hua (Zbl **51**, 280).

Vinogradov, A. I. On some new theorems of the additive theory of numbers. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 875–876. (Russian)

The author announces the following result, closely related to one of Linnik [Mat. Sb. N.S. **32(74)** (1953), 3–60; MR **15**, 602]. Theorem 1: Let $p>2$ be a given integer, and let any sufficiently large number N be written in the scale of p as

$$N=a_0+a_1p+\cdots+a_Rp^R.$$

Then it is sufficient to diminish by 1 at most k of the coefficients $a_0, \cdots, a_R$ (where k depends only on p) in order to obtain a number representable as the sum of two primes. Another more complicated result (Theorem 2), which implies Theorem 1, is enunciated, as is also the main lemma on which the proof of Theorem 2 is said to be based.

H. Davenport (London).

See also: Rédei, p. 343; Lamprecht, p. 350.

Algebraic Number Theory

Dufresnoy, J., et Pisot, Ch. Etude de certaines fonctions méromorphes bornées sur le cercle unité. Application à un ensemble fermé d'entiers algébriques. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 69–92.

Let S be the set of algebraic integers $\theta>1$ all of whose conjugates are interior to the unit circle. Then S was shown to be closed by R. Salem [Duke Math. J. **11** (1944), 103–108; MR **5**, 254] with a conjecture on the smallest element θ_2, established by C. L. Siegel [ibid. **11** (1944), 597–602; MR **6**, 39] with a conjecture on the smallest limit point θ_∞ of S, established by the authors [Ann. Sci. Ecole Norm. Sup. (3) **70** (1953), 105–133; MR **15**, 605]. The authors give a detailed description of S up to $\hat{\theta}_{15}$ $(>\theta_\infty)$ obtaining (1) the ascending sequence θ_n $(n\geq 2)$ defined as a root of $(1-z^nh(z))/(1-z)$ for n even and $(1-z^nh(z))/(1-z^2)$ for n odd; (2) the ascending sequence θ_n' $(n\geq 1)$ root of $1-z^2+z^nh(z)$; (3) the descending sequence $\hat{\theta}_n$ $(n\geq 15)$ root of $[1+z^nh(z)]/(1+z)$ for n even and $1+z^nh(z)$ for n odd; (4) the descending sequence $\hat{\theta}_n'$ $(n\geq 17)$ root of $1-z^2-z^nh(z)$; (5) $\theta_\infty=(5^{\frac{1}{2}}+1)/2$ limit point of the sequences; and finally (6) the root of $1-z+z^3-z^4+2z^5-z^6$. Here $h(z)=1+z-z^2$. The connection with the coefficient problem of bounded functions is established for instance by defining $f(z)=P(z)/[\pm z^nP(1/z)]$, where $P(z)=\pm z^n+\cdots$ defines θ; then $|f(z)|=1$ on the unit circle but $f(z)$ has a pole at $1/\theta$. Now suppose $f(z)$ is known to begin with $1+z+z^2+\cdots$; then with

$$\lambda(z)=(z^2-z-1)/(z^2-1),$$

the authors define $g(z)=[\lambda(1/z)f(z)-1]/[\lambda(z)-f(z)]$ so that $g(z)$ is holomorphic in the unit circle, $|g(z)|\leq 1$ on the boundary, and $g(z)$ has integral coefficients; whence $g(z)$ is $\pm z^n$, determining $f(z)$, $P(z)$, and thus θ $(=\theta_n'$ or $\hat{\theta}_n')$.

Harvey Cohn (Detroit, Mich.).

Butts, Hubert, Hall, Marshall, Jr., and Mann, H. B. On integral closure. Canad. J. Math. **6** (1954), 471–473.

The authors make the following two remarks. (1) Let J be an integral domain with unit element, F its quotient field. Assume that J is integrally closed. If $p(x)=ax^m+\cdots+a_m\in J[x]$ is reducible in $F[x]$, then $a\cdot p(x)$ is reducible in $J[x]$. (2) If J is integrally closed, then $J[x]$ is integrally closed.

Y. Kawada (Tokyo).

Mann, H. B. On an exceptional phenomenon in certain quadratic extensions. Canad. J. Math. **6** (1954), 474–476.

Let Σ be an algebraic number field and Ω/Σ be a cyclic extension of a prime degree l. Then Ω would contain a unit ξ such that $\xi^{\nu^\beta}\in\Sigma$, $\xi^{\nu^{\beta-1}}\notin\Sigma$, $\beta>1$. This is possible only for $l=2$ and the author determines all possible cases.

Y. Kawada (Tokyo).

Chao, Nieh Ling. Discrete-valued complete fields with residue class fields of characteristic p. J. Chinese Math. Soc. (N.S.) **1** (1951), 377–394. (Chinese summary)

Using some of the methods employed by O. Teichmueller [J. Reine Angew. Math. **176** (1936), 141–152] and S. MacLane [Ann. of Math. (2) **40** (1939), 423–442] for the existence and description (to within analytic isomorphisms) of all fields complete with respect to a discrete rank-one valuation with prescribed residue class field of prime characteristic, the author modifies suitably the original approach of H. Hasse and F. K. Schmidt [J. Reine Angew. Math. **170** (1933), 4–63] which uses the tool of "ground rings", in order to furnish a correction to their proof.

O. F. G. Schilling (Chicago, Ill.).

Uchiyama, Saburô. Note on the mean value of $V(f)$. II. Proc. Japan Acad. **31** (1955), 321–323.

Let $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x$, $a_i\in GF(q)$, $q=p^r$,

$1<n<p$, and let $V(f)$ denote the number of distinct values of $f(x)$, $x \in GF(q)$. The reviewer [same Proc. **31** (1955), 119–120; MR **17**, 130] showed that

$$\sum_{a_1} V(f) \geqq \frac{q^2}{2q-1} > \frac{q^2}{2},$$

while the present writer [ibid. **31** (1955), 199–201; MR **17**, 130] showed that

$$\sum_{\deg f=n} V(f) = \sum_{r=1}^{n} (-1)^{r-1} \binom{q}{r} q^{n-r} = c_n q^n + O(q^{n-1}) \quad (q \to \infty),$$

where

$$c_n = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1} \frac{1}{n!}.$$

The present paper contains a proof of the following theorem:

$$\sum_{a_1, \cdots, a_{n-r-1}} V(f) = q^r \sum_{\deg f=n} V(f) + R_{n,r},$$

where

$$R_{n,r} = \begin{cases} 0 & (r=1) \\ O(q^{\theta n}) & (r \geqq 2) \end{cases}$$

with $\theta = 1 - r^{-1}$. In particular, if $n \geqq r(r+1)$, then

$$\sum_{a_1, \cdots, a_{n-r-1}} V(f) = c_n q^{n-r} + O(q^{n-r-1}).$$

The proof depends on some previous results of the author [ibid. **30** (1954), 523–527; **31** (1955), 267–269; MR **16**, 570; **17**, 131]. It is remarked that analogous results can be obtained for the sum

$$\sum_{a_{t+1}, \cdots, a_{n-r-1}} V(f) \quad (r+t \geqq 2).$$

L. Carlitz (Durham, N.C.).

Eda, Yoshikazu. On the canonical basis of ideals. Sci. Rep. Kanazawa Univ. **2** (1953), no. 1, 15–21.

P, I, K and $\mathfrak{o}$ denote the rational number field, the integral domain of P, the algebraic number field over P with degree n and the integral domain of K, respectively. Let $\omega_1, \omega_2, \cdots, \omega_n$ be a basis of $\mathfrak{o}$ and $\omega_i \omega_j = \sum_{k=1}^{n} \tau_{ij}^{k} \omega_k$, $\tau_{ij}^{k} \in I$. It is well-known that a basis of an ideal $\mathfrak{o}$ in K has a basis $a_1, a_2, \cdots, a_n$, with $a_i = a_i^1 \omega_1 + \cdots + a_i^i \omega_i$, $a_j^i \in I$, $N\mathfrak{o} = \prod_{i=1}^{n} a_i^i$, where $N\mathfrak{o}$ is the norm of $\mathfrak{o}$. Let $\Delta_{kj}^{m} = [c_{\nu\lambda}]$ be a matrix, where $c_{\lambda 1} = \tau_{kj}^{m+\lambda-1}$ $(\lambda = 1, \cdots, n-m+1)$, $c_{\nu 1} = a_{m+\nu-1}^{m}$ $(\nu = 2, \cdots, n-m+1)$, $c_{\nu\lambda} = a_{m+\nu-1}^{m+\lambda-1}$ $(\lambda = 2, \cdots, n-m+1, \nu = 2, \cdots)$ $(a_\nu^\lambda = 0$ for $\lambda > \nu)$. It is proved that the bases $(a_1, a_2, \cdots, a_n)$ may be choosen such that it also has the property $\sum_{k=1}^{i} a_i^k \Delta_{kj}^{m} = 0$ mod $\prod_{i=m}^{n} a_i^i$ $(i, j, k = 1, 2, \cdots, m)$. This basis is called canonical. *H. Bergström* (Göteborg).

Eda, Yoshikazu. A note on the general divisor problem. Sci. Rep. Kanazawa Univ. **3** (1955), no. 1, 5–9.

Let Ω be an algebraic extension of the rational number field of degree 1 and K an extension of Ω of degree m and let $T_\kappa(n)$ denote the number of ways of expressing an integral ideal n in Ω as a product of κ ideal factors in K $(\kappa \geqq 2)$. Consider the sum $A_\kappa(x) = \sum_{N(n) \leqq x} T_\kappa(n)$. By a well-known theorem $A_\kappa(x)$ may be given as a sum of a polynomial in $\log x$ and a remainder term. The author proves two theorems by which he can determine the polynomial in the case of an Abelian extension K over the rational number field (and for $l=1$). *H. Bergström.*

Lamprecht, Erich. Arithmetische Zetafunktionen zu zyklischen p-Körpern von zwei Veränderlichen über einem Galoisfeld. Arch. Math. **6** (1955), 266–274.

In this note the author studies generalizations of the zeta function to fields of Kroneckerian dimension 2 and prime characteristic p. For the definition of his zeta functions the author employs prolongations of valuations on subfields of Kroneckerian dimension 1 by means of Gauss' Lemma. Thus his functions depend on the fibering of the given field (problems which do not arise for zeta functions of function fields of one variable over a finite algebraic number field indicate that his definition is bound to lead to shortcomings of his approach), and obvious difficulties concerning the dependence of the zeta function on the particular fibering immediately arise. In order to secure his tentative results the author makes restrictions on the generation of his fields as p-extensions of rational function fields, so that the ramification theory may lead to simplifications of Gaussian sums involved in the discussion of his functions. The lack of generally valid functional equations (or modifications thereof) of his zeta functions indicates the tentative nature of the author's approach. [For other studies on the zeta function of fields of prime characteristic see, e.g., A. Weil, Bull. Amer. Math. Soc. **55** (1949), 497–508; MR **10**, 592; and S. Lang and A. Weil, Amer. J. Math. **76** (1954), 819–827; MR **16**, 398.] *O. F. G. Schilling.*

Byers, G. Cleaves. Class number relations for quadratic forms over GF $[q, x]$. Duke Math. J. **21** (1954), 445–461.

The author develops the general theory of binary quadratic forms with coefficients in $GF(q)[x]$. One of the main results is the following. Let $h(\Delta)$ denote the number of classes of equivalent binary quadratic forms of discriminant Δ. Then for any given element ∇ in $GF(q)[x]$ of degree $2m+1$ the relation $\sum_R h(R-\nabla) = 2\sum_K q^k$ holds, where $\sum_R$ extends over all $R \in GF(q)[x]$ with $\deg R \leqq m$ and $\sum_K$ extends over all primary factors K of ∇ with $k = \deg K > m$. *Y. Kawada* (Tokyo).

See also: Selfridge, Nicol and Vandiver, p. 348.

Geometry of Numbers, Diophantine Approximation

Ehrhart, Eugène. Propriétés arithmo-géométriques des ovales. C. R. Acad. Sci. Paris **241** (1955), 274–276.

This is a continuation of three previous notes [same C. R. **240** (1955), 483–485, 583–585, 935–938; MR **16**, 574, 740, 908]. Let K denote a plane non-central convex body (i.e. "oval") with area >6 and let Λ denote any lattice of determinant 1 having a point at the centroid G of K. Then, under certain restrictions which are probably spurious, it is shown that K contains a point of Λ, other than G, together with its image in G. An example in which K is a triangle suffices to show that the constant 6 is best possible. *J. H. H. Chalk* (London).

Ehrhart, Eugène. Propriétés arithmogéométriques des polygones. C. R. Acad. Sci. Paris **241** (1955), 686–689.

The critical cases in the author's extension of Minkowski's fundamental theorem [same C. R. **240** (1955), 483–485; MR **16**, 574] are characterized. *J. H. H. Chalk* (London).

Oliwa, Godfried. Eine Anwendung des Übertragungsprinzips von Hlawka. Anz. Österreich. Akad. Wiss. Math.-Nat. Kl. **1953**, 239–242.

Verf. folgert unmittelbar aus dem inhomogenen Linear-

formensatz mit der Schranke von Tschebotarev: Besitzt das System der Kongruenzen $Ay_i \equiv a_i \pmod{m_i}$ ($i=0, 1, 2, \cdots, n$), ($A, a_0, \cdots, a_n, m_0, \cdots, m_n$ ganz) eine Lösung, dann ist für jedes System reeller Zahlen $\xi_1, \cdots, \xi_n$

$$|z_0||\xi_1 z_0 - z_1| \cdots |\xi_n z_0 - z_n| \leqq 2^{-(n+1)/2} m_0 \cdots m_n$$

in ganzen Zahlen $z_0, z_1, \cdots, z_n$ mit $Az_i \equiv a_i (m_i)$ ($i=0, 1, \cdots, n$) lösbar. [Bem. des Ref.: Die Formulierung des Satzes in der Note des Verf. ist richtig, wenn $a_0 \not\equiv 0 \pmod{m_0}$.] Verf. gibt noch eine Verallgemeinerung auf imaginärquadratische Zahlkörper.

E. Hlawka (Zbl 52, 281).

Sawyer, D. B. On the covering of lattice points by convex regions. II. Quart. J. Math. Oxford Ser. (2) **6** (1955), 207–212.

The author shows that the theorem of the first part of the paper [same J. (2) **4** (1953), 284–292; MR **15**, 607] remains true if the condition that the set K is closed and central is omitted entirely. This result was conjectured by J. L. Massera and J. J. Schäffer [Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadíst. **2** (1951), 55–74; MR **13**, 768]. It is pointed out in a note that the same result has been obtained independently by Shäffer, whose work is in course of publication. *N. G. de Bruijn.*

Block, Daniel. Some properties of the convergents of $\sqrt{2}$. Scripta Math. **21** (1955), 208–213.

ANALYSIS

★**Menger, Karl. Calculus, a modern approach.** Ginn and Company, New York, 1955. xviii+354 pp.

A preliminary mimeographed version has already been reviewed [MR **16**, 575].

Lowan, Arnold N. Note on an elementary method for generating inequalities. Scripta Math. **21** (1955), 218–220.

Schoenberg, I. J. On the zeros of the generating functions of multiply positive sequences and functions. Ann. of Math. (2) **62** (1955), 447–471.

The sequence $\{a(n)\}_{-\infty}^{\infty}$ ($a(n) \geqq 0$, $\sum_{-\infty}^{\infty} a(n) < \infty$) is said to be k-positive if $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_n$ implies that $\det [a(i_\nu - j_\mu)]_{1,n} \geqq 0$ for $n=1, 2, \cdots, k$. If P_k denotes the class of k-positive sequences, then evidently $P_1 \supset P_2 \supset \cdots \supset P_k \supset \cdots$. A complete characterization of the sequences $\{a(n)\}_{-\infty}^{\infty}$ belonging to P_∞ in terms of the structure of the function $\sum_{-\infty}^{\infty} a(n) z^n$ has been obtained by the author [J. Analyse Math. **1** (1951), 331–374; MR **13**, 923] and by Edrei [Trans. Amer. Math. Soc. **74** (1953), 367–383; MR **14**, 853]. In the present paper a similar investigation is initiated for the classes P_k. Thus it is proved that if $\cdots, 0, 0, a(0), a(1), \cdots, a(m), 0, 0, \cdots$, where $a(0)>0$, $a(m)>0$, belongs to P_k then $a(0)+a(1)z+a(2)z^2+\cdots+a(m)z^m$ has no zeros in the sector $|\arg z| < k\pi/(m+k-1)$ and the constant on the right cannot be improved, a fact which follows from a consideration of the polynomial $\sum_{\nu=0}^{m} \left\{ {m \atop \nu} \right\} z^\nu$ where

$$\left\{ {m \atop \nu} \right\} =$$

$$\sin m\theta \sin (m-1)\theta \cdots \sin (m-\nu+1)\theta / \sin \theta \sin 2\theta \cdots \sin \nu\theta.$$

A function $f(x)$ defined for $-\infty < x < \infty$ and such that $f(x)$ is measurable, $f(x) \geqq 0$, $\int_{-\infty}^{\infty} f(x)dx < \infty$, is said to be k-positive if $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$ implies that $\det [f(x_i - y_j)]_{1,n} \geqq 0$ for $n=1, 2, \cdots, k$. As before let P_k be the class of k-positive functions. Using the previously described result on k-positive sequences the author shows that if $f(x)$ belongs to P_k and if $f(x)=0$ for $|x| \geqq \pi/2$ then $\int_{-\infty}^{\infty} f(x) e^{-sx} dx$ has no zero in the strip $|\mathrm{Im}\, s| < k$, and the constant k cannot be improved. Here the extremal function is $\cos^\alpha x$ for $|x| \leqq \pi/2$ and 0 for $x > \pi/2$, which is k-positive if $\alpha \geqq k-2$. *I. I. Hirschman.*

Mammana, Carmelo. Sul problema algebrico dei momenti. Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 133–140 (1955).

The author considers the classical problem: find n points $x_1 < \cdots < x_n$ on the real line and n positive numbers p_i, such that the moments $\sum_{i=0}^{n} p_i x_i^k$ have prescribed values μ_k ($k=0, \cdots, 2n-1$). In the first part of the paper the classical condition for the existence of solution is derived by a method used by Ghizzetti in the trigonometrical case. In the second part the problem is considered under the further requirement $x_1 \geqq a$ ($x_n \leqq b$). [Reviewer's comment: The proof, but not the result, is new; cf. M. Krein, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 4(44), 3–120; MR **13**, 445.] *M. Cotlar* (Mendoza).

Šilov, G. E. On a problem of quasi-analyticity. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 893–895. (Russian)

L'auteur se pose le problème suivant: quelle condition faut-il imposer aux quantités M_{kp} ($k, p=0, 1, 2, \cdots$) pour que de

$$|x^k \varphi^{(p)}(x)| \leqq C A^k B^p M_{kp} \quad (k, p \geqq 0, \ -\infty < x < \infty),$$

(φ étant indéfiniment dérivable) résulte que $\varphi \equiv 0$. L'auteur résout le problème pour le cas où $M_{kp} = K^{k^\alpha} p^{p^\beta}$ ($\alpha, \beta \geqq 0$). La classe correspondante est désignée par S_α^β. Une condition nécessaire et suffisante pour que S_α^β ne contienne que la fonction $\varphi \equiv 0$ est que les quantités α, β satisfassent à une des trois conditions suivantes: a) $\alpha=0$, $\beta \leqq 1$; b) $\alpha \leqq 1$, $\beta=0$; c) $\alpha>0$, $\beta>0$, $\alpha+\beta<1$.

S. Mandelbrojt (Paris).

Theory of Sets, Functions of Real Variables

★**Kamke, E. Mengenlehre. 3te Aufl.** Sammlung Göschen Band 999/999a. Walter de Gruyter & Co., Berlin, 1955. 194 pp. DM 4.80.

The 2d edition was published in 1947 [MR **9**, 573], the 1st edition in 1928. Although the general plan is the same, this edition has undergone some revision and expansion.

Zakon, E. On fractions of ordinal numbers. Technion. Israel Inst. Tech. Sci. Publ. **6** (1954/5), 94–103. (Hebrew summary)

Let $p = \alpha/\beta = (\mu_1, \mu_2, \cdots, \mu_n, \infty, \infty, \cdots)$ be an ordinal fraction [Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 187]; the author defines the grade of p to be the smallest ordinal ν such that $\mu_k < \omega^{\nu+1}$ ($k=1, \cdots, n$). Main results: if p and r are each of grade $\leqq \nu$, with $p < r$, then there exist $|\omega^{\nu+1}|$ fractions m, resp. q, resp. s, of grade ν, such that $m < p < q < r < s$; if $\nu < \sigma$, then

there are $|\omega^{\sigma+1}|$ fractions $<$, resp. $>$, all fractions of grade $\leqq\nu$. The author promises to apply these results in a later paper. [Misprints that might cause confusion: γ/α is inverted, p. 98, lines 2 and 17b; $<$ is reversed, p. 100, line 10b.] *L. Gillman* (Lafayette, Ind.).

Ohkuma, Tadashi. Sur quelques relations concernant les opérations P_α et S_α sur les classes d'ensembles. Proc. Japan Acad. **31** (1955), 410–413.

Announcement (the proofs to appear later) of generalizations of results of Sierpiński and Tarski [Fund. Math. **15** (1930), 292–300] and Koźniewski and Lindenbaum [ibid. **30** (1930), 342–355]. Let $\pi_\alpha(\beta)$ denote the smallest ordinal γ such that $\prod m_\lambda<\aleph_\gamma$ for every family (m_λ) of $<\aleph_\alpha$ cardinals each $<\aleph_\beta$. The two main results are the following. Theorem 4: $P_\alpha S_\beta P_\gamma=P_\delta S_\varepsilon$ if and only if $\beta=\varepsilon$, and either (i) $\pi_\beta(\gamma)<\alpha=\delta$, or (ii) $\pi_\beta(\gamma)=\mathrm{cf}(\alpha)=\alpha=\delta$, or (iii) $\alpha\leqq\beta=\gamma=\delta$ and β is strongly inaccessible. Theorem 6: Assuming the generalized continuum hypothesis, then $S_\alpha P_\beta S_\gamma=P_\delta S_\varepsilon$ if and only if either (i) $\pi_\alpha(\beta)=\beta=\delta$, $\gamma=\varepsilon$, and $\alpha\leqq\mathrm{cf}(\gamma)$, or (ii) $\gamma\leqq\delta=\varepsilon=\alpha=\beta$ and β is (strongly) inaccessible. *L. Gillman* (Lafayette, Ind.).

Ribeiro Albuquerque, José. Finite sets. Gaz. Mat., Lisboa **15** (1955), no. 60–61, 24–28. (Portuguese)

A discussion of finite and infinite sets, the definition of an infinite set given here being a variation of (but equivalent to) that due to Dedekind. *S. Ginsburg.*

Zaubek, Othmar. Ein Beitrag zum Borelschen Überdeckungssatze. Arch. Math. **6** (1955), 444–447.

The basic theorem is: If A is a subset of the sum of a sequence of sets $V_1, \cdots, V_n, \cdots$, then A is a subset of the sum of a finite number of these sets if and only if, for every subset U of A, there exists a V_k such that intersection UV_k contains infinitely many elements. This leads to a generalized Borel theorem of denumerable type: If A is a subset of a general topological space, which is covered by the sequence of sets $V_1, \cdots, V_n, \cdots$, and if every infinite subset of A has at least one limiting point which is an interior point of the sum of a finite number of V_k, then A is covered by a finite number of the sets V. Here a general topological space is one in which there is a class of "open" sets, h being a limiting point of A if every "open" set containing h contains infinitely many elements of A. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Greco, Donato. Criteri di compattezza per certe classi di funzioni in n variabili. Ricerche Mat. **3** (1954), 220–246.

Let T be a domain in S_n whose frontier satisfies certain regularity hypotheses, $\{u(P)\}$ a set of functions of n variables all of class C^p, uniformly bounded in T, satisfying

$$\sum_{k\leqq p}\int_{S\cap T}\varrho^{mk-n}\sum_{k_1+\cdots+k_n=k}\left|\frac{\partial^k u}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\right|^m d\tau\leqq M$$

for every sphere S, where $mp>n-1$; suppose that there are two functions $F_1(P,z)$ and $F_2(P,z)$, increasing in z and defined for $P\in T$ and z on the (finite) range of values of $u(P)$ in T, and continuous in both variables, such that there is no relative maximum to $F_1(P,u(P))$ interior to T and no relative minimum for $F_2(P,u(P))$ interior to T. Then the family $u(P)$ is equicontinuous in every region of T at a positive distance from the frontier of T.

In a previous paper [Ricerche Mat. **1** (1952), 124–144; MR **14**, 29] he had $mp>n$ (case $\nu=0$) and did not require the functions F_1 and F_2 to exist. The present result in two dimensions, and the functions F_1 and F_2 in that case, is due to Caccioppoli [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) **22** (1935), 305–310] and Leray [J. Math. Pures Appl. (9) **17** (1938), 89–104]. *J. M. Danskin.*

Chamberlin, R. E. Remark on the averages of real functions. Pacific J. Math. **5** (1955), 663–668.

Let $f(x)$ be a real continuous function defined for $a<x<b$ and let

$$L(f;x,t)\equiv\frac{f(x+t)+f(x-t)}{2}-\frac{1}{2t}\int_{x-t}^{x+t}f(s)ds.$$

The author exhibits a continuously differentiable nonlinear function $f(x)$ such that for each $x\in(a,b)$, there exist infinitely many values of t for which $L(t;x,t)=0$. This result complements results due to Walsh, and Beckenbach and Reade [Trans. Amer. Math. Soc. **53** (1943), 230–238; MR **4**, 199]; it also casts suspicion on a "theorem", stated without proof, by Reade [Duke Math. J. **10** (1943), 531–536; MR **5**, 7] and the extension of that "theorem" by Aki [Kodai Math. Sem. Rep. **1950**, 11–12; MR **12**, 258]. *M. O. Reade* (Ann Arbor, Mich.).

See also: Ghika, p. 385; Schaefer, p. 390; de Groot, p. 393.

Theory of Measure and Integration

Erdös, Paul, and Oxtoby, John C. Partitions of the plane into sets having positive measure in every non-null measurable product set. Trans. Amer. Math. Soc. **79** (1955), 91–102.

Let X and Y be measure spaces; a (measurable) set $E\subset X\times Y$ is here said to have "property M" if $\mu(E\cap(A\times B))>0$ whenever A, B are subsets of X, Y, respectively, of positive measure. D. Maharam and the reviewer have shown (unpublished) that the plane, regarded as the product of two lines, can be partitioned into two disjoint sets each of which has this property, and have asked whether such partitions exist which retain the property after arbitrary rotations. The authors show that this is so; in fact (a special case of Theorem 4) there exists a partitioning of the plane into a sequence of disjoint sets E_i such that, for every 1-1 mapping ϕ of the plane onto itself of class C^2 and non-vanishing Jacobian, the sets $\phi(E_i)$ all have property M. The construction depends on the following generalization of Steinhaus's theorem: if $u(x,y)$ is a real-valued function of class C^1, and $u_x u_y=0$ on a null set at most, and if D is a metrically dense set of real numbers, then the set $\{(x,y)|u(x,y)\in D\}$ has property M. On the other hand, the group of permissible transformations cannot include the group H of all area-preserving homeomorphisms of the plane onto itself, for the authors show that to any plane set E of positive measure there corresponds $h\in H$ for which $h(E)$ contains a non-null product set $A\times B$. They show further that all but a first-category family of plane sets E are such that both E and its complement have property M. Finally, when X and Y are arbitrary σ-finite measure spaces, it is shown that for $X\times Y$ to admit a partitioning into two (or $\aleph_0$) disjoint sets with property M, it is necessary and sufficient that the measure algebras of X and Y be non-atomic. *A. H. Stone* (Manchester).

Dowker, Yael Naim. On measurable transformations in finite measure spaces. Ann. of Math. (2) **62** (1955), 504–516.

Let T be a 1-to-1 measurable, non-singular transformation of a measure space (S, m) with $m(S)=1$. Let $m_n(A)=\sum_1^n m(T^iA)/n$, and let $\mu\sim m$ signify that μ has the same nullsets as m. (1) There exists a finite, invariant $\mu\sim m$ if and only if $m(A)>0$ implies $\liminf m(T^nA)>0$. (2) S can be decomposed into invariant sets S_1 and S_2 such that S_2 admits a finite, invariant $\mu\sim m$ and S_1 contains a set A with $\liminf m(T^nA)=0$ and

$$\limsup m(T^nA)=m(S_1).$$

(3) If T is metrically transitive and m is non-atomic then $\mu_n(A)-m_n(A)\to 0$ for every measurable set A and for every $\mu\sim m$ with $\mu(S)=1$. (4) If T is metrically transitive and m is non-atomic, and if there exists a σ-finite, invariant $\mu\sim m$ with $\mu(S)=\infty$, then for any $0\leq\alpha\leq\beta\leq 1$ there is a set A with $\liminf m_n(A)=\alpha$ and $\limsup m_n(A)=\beta$. In a note added in proof it is stated, without proof, that (5) if there is no finite invariant $\mu\sim m$ then $m_n(A)\to 0$ for some A with $m(A)>0$, and that the last hypothesis of (4) can be replaced by that of (5). *J. C. Oxtoby.*

Bauer, Heinz. Zur Theorie des Riemann-Integrals in lokal kompakten Räumen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1955**, 187–208.

The author first summarizes Loomis's theory of Riemann integration and content for vector lattices of functions [Amer. J. Math. **76** (1954), 168–182; MR **15**, 631]. He then compares this with the theory of Riemann integration and content on locally compact Hausdorff spaces announced by Haupt and Pauc [C.R. Acad. Sci. Paris **230** (1950), 711–712; MR **11**, 587], and shows that the former theory can be subsumed under the latter. The paper ends with a discussion of vague convergence of functionals and the behavior of the corresponding contents.

E. Hewitt (Princeton, N.J.).

Nicolescu, Lilly. Intégrales Perron-Stieltjès. Com. Acad. R. P. Romîne **4** (1954), 555–559. (Romanian. Russian and French summaries)

Announcement of the results of the paper reviewed below. *M. Cotlar* (Mendoza).

Nicolescu, Lilly-Jeanne. Intégrales Perron-Stieltjès. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6** (1954), 755–770. (Romanian. Russian and French summaries)

α being a monotone-increasing function in $[a, b]$, $f(x)$ is said to be P^*S-integrable with respect to α if given $\varepsilon>0$ there exist four functions φ^i $(i=1, 2, 3, 4)$ such that: 1) φ^i is attached to α; 2) $\varphi^i(a)=0$; 3) $|\varphi^i(x)-\varphi^j(x)|<\varepsilon$ $(i, j=1, 2, 3, 4)$; 4) for every $x\in[a, b]$, $-\infty\neq\Delta_+\varphi^1(x)\geq f(x)$, $+\infty\neq\Delta_-\varphi^2(x)\geq f(x)$, $+\infty\neq\Delta^-\varphi^3(x)\leq f(x)$, $-\infty\neq\Delta^+\varphi^4(x)\leq f(x)$, except at a denumerable set of points of continuity of α, where for instance $\Delta^+\varphi$ is defined as follows: $\Delta^+\varphi(x)=D^+\varphi(x)$ if $\alpha(x-0)=\alpha(x)$ or $x=a$, and $\Delta^+\varphi(x)=D^-\varphi(x)$ if $\alpha(x-0)\neq\alpha(x)$ or $x=b$. The author proves the equivalence of this definition to Ridder's definition of Perron-Stieltjes integral, and from this equivalence he deduces that if $f\in P^*S$, α is continuous, and g of bounded variation, then $fg\in P^*S$ and the theorem of integration by parts holds. *M. Cotlar* (Mendoza).

Denjoy, Arnaud. Totalisation des dérivées premières généralisées. I. C. R. Acad. Sci. Paris **241** (1955), 617–620.

F being continuous on an interval,

$$f(x)=D_gF(x)=\lim_{h=0}\frac{F(x+h)-F(x-h)}{h}$$

denotes, when it exists, the first generalized (symmetric) derivative of F; if $G(x)$ is the primitive of $F(x)$, then $f(x)$ is the second generalized derivative $D_{g^2}G$ of G. The author makes use of some of his concepts and notations of his series of books (D) [Leçons sur le calcul des coefficients d'une série trigonométrique; t. I–IV, Gauthier-Villars, Paris, 1941, 1949; MR **8**, 260; **11**, 99], in which a totalisation $(T_{g^2})_0$ was introduced, entailing nine operations and enabling the calculation of the primitive G (except for a linear function) from $D_{g^2}G$. This enables obtaining F from f; G may be without a derivative on a non-denumerable everywhere-dense set; however, simplifications arise due to the fact that $G'(x)=F(x)$ is in the present case continuous. The problem undertaken here and completed in a later note is to find a symmetric (first-order) totalisation T_g, which constructs the variation $V(F, x, x')=F(x')-F(x)=T_g(f, x, x')$ for all x, x' on segment ab ($F(a+0)$, $F(b-0)$ assumed to exist). Let

$$A(x)=\max_{|h|>0}\left|\frac{F(x+h)-F(x-h)}{2h}\right|.$$

(I) $A(x)$ is bounded and hence f is summable on every portion $\bar{\omega}(P)$ of any perfect set P, disjoint of a closed set $K(P)$ nondense on P, $K(P)$ being the set of points of unboundedness (on P) of $A(x)$; the first operation is L (Lebesgue)-integration on perfect sets, on which f is summable (II). The second operation is passage to the limit to obtain $V(x, x')$ on $\bar{e}$, when $V(x, x')$ is known on a set e. (III) Let $E(A)$ be the set where $A(x)\leq A$. On using developments of the type found in (D; p. 257–262) and of concepts such as index of a set and coefficient of isolation, it is shown that a.e. (almost everywhere) on $E(A)$ the four derivates of $F(x)$ are finite; whence $f=F'$ a.e. on ab. If the index of P is finite at all points, then the four derivates of $F(x)$ in $P-K(P)$ are finite; if $K(P)=0$, then $f(x)$ is completely totalisable on P (definition here conforms with author's earlier use of the term). If $K(P)\neq 0$, there exists a nondense subset of P, so that $V(x, x')=T_g(f, x, x')$ can be calculated by the third operation of the complete totalisation (also that of T_g) on every segment xx' disjoint of this subset. *W. J. Trjitzinsky.*

Denjoy, Arnaud. Totalisation des dérivées premières symétriques. II. Intégrale de Lebesgue et mesure borélienne. C. R. Acad. Sci. Paris **241** (1955), 829–832.

In this note the author completes the theory of the totalisation T_g involved in the paper reviewed above. Let $P\subset A$, with extremities α, β, be perfect and nondense, and suppose that the contiguous problem for P has been solved. One is to find $V(x, x')$ on $\alpha\beta-P$; this will yield $V(x, x')$ on $\alpha\beta$ (by the second operation). One needs only to examine: the case (1°), when P is of the second kind $(P=\Omega'(P))$ which leads to use of primary segments of suitably great coefficient and to a limiting process, termed the fourth operation of T_g; the case (2°), when P is of the third kind, leading to the fifth operation of T_g, involving the first three operations and a passage to the limit, as $\lambda\to+\infty$. In the latter connection, an infinity of

primary segments ϱ_λ (of coefficient $>\lambda$) covers P, and x and x' are eventually outside $I(P,\lambda)+\sum\varrho_\lambda$ ($f(x)$ is completely totalisable on $I(P,\lambda)$). (V) P being any perfect set, let $H(P)$ be the set (nondense) of points of P not in any isolated portions of the definite kind. Let $\{\overline{\omega}_n\}$ be a sequence of portions of P, of diameter tending to zero, each point of P being interior to an infinity of the $\overline{\omega}_n$; on every $\overline{\omega}_n$ in a portion $\overline{\omega}$ the third, fourth and fifth operations apply. The problem contiguous to a closed set $S(P)\subset P$, nondense in P, is solved. One is thus lead to a well defined and specific transfinite process, which constitutes the totalisation $T_s(f,x,x')$ on ab. In the remainder of the note $F(x)$ is merely measurable. It is proved that the set of points, where the first symmetric derivative is not the normal derivative, has length zero. Let $f(x)>0$ be given on a set E. If the L-integral $I=\int_E f(x)dx$ exists, then $I=J=\int_0^\infty l(y)dy$ (in the sense of Riemann), where $l(y)$ is the Borel length of the set $s(y)=[x\in E, f(x)\leqq y]$. The Borel area $\alpha(S)$ of $S[0<y<f(x), x\in E]$ exists and equals I, if I exists. A proof is given of the fact that, if $\alpha(S)$ exists, then $\alpha(S)=J$ $(=I)$.

W. J. Trjitzinsky (Urbana, Ill.).

König, Heinz. Multiplikation und Variablentransformation in der Theorie der Distributionen. Arch. Math. **6** (1955), 391–396.

A vector space $E=\{\alpha,\beta,\cdots,f,g,\cdots\}$ is assumed to satisfy the following axioms (Schwartz's distributions will be a particular case). I: E contains all continuous functions $f(x)$; a linear operator α' is defined for all α in E, coinciding with the derivative if $\alpha=f$ has a continuous derivative. II: A product $\alpha\circ\beta$, valued in E, linear in each of α, β, is defined for some α,β coinciding with the ordinary product if α, β are both continuous functions; $f\circ\alpha$ is defined for all continuous functions f, arbitrary α; if $\alpha\circ\beta$ is defined, so are $(f\circ\alpha)\circ\beta$ and $\alpha\circ(f\circ\beta)$; $1\circ\alpha=\alpha$ for all α; $(\alpha\circ\beta)'$ is defined and equals $(\alpha'\circ\beta)+(\alpha\circ\beta')$ whenever either of the last two products is defined. III: If Φ, Ψ are inverse (1, 1) mappings of the real axis on itself, each possessing derivatives of all orders, then inverse mappings $\overline{\Phi}$, $\overline{\Psi}$ are defined on E such that $(\overline{\Phi}f)(x)=f(\Phi(x))$ for continuous functions f; if $\alpha\circ\beta$ is defined, then $\overline{\Phi}(\alpha\circ\beta)=(\overline{\Phi}\alpha)\circ(\overline{\Phi}\beta)$; $(\Phi\alpha)'=\Phi'\circ(\overline{\Phi}\alpha')$ for all α in E.

Now suppose δ in E satisfies $x\circ\delta=0$. The author proves that δ must be 0 if $f'\circ\delta$ is defined for all continuous f. In particular, multiplication cannot be extended to all distributions of Schwartz, if axioms I, II, III are to remain satisfied.

Consider the formula $f\circ(\alpha\circ\beta)'=((f\circ\alpha')\circ\beta)+(\alpha\circ(f\circ\beta'))$ for all α, β for which $\alpha\circ\beta'$ is defined. The author deduces this formula for all f of the form $1/\Phi'$ with Φ as in Axiom III (necessarily $\Phi'(x)\neq 0$ for all x). He represents x^{2m} in the form

$$\frac{1}{\Phi_1'(x)}+\frac{1}{\Phi_2'(x)}-\frac{1}{\Phi_3'(x)}$$

(this is the essential part of the proof) and so verifies the formula for $f(x)=x^{2m}$, $f(x)=(1+x)^{2m}$ and finally for f an arbitrary polynomial (in the proof only the cases $f(x)=x$ and $f(x)=x^2$ are actually used). Setting $\alpha_0=(x(\log|x|-1))\circ\delta$ he shows that $(x\circ\alpha_0)'=\delta$, $x\circ\alpha_0=0$ and hence $\delta=0$.

I. Halperin (Kingston, Ont.).

Ishihara, Tadashige. Note on an extension of multiplication of distributions. Proc. Japan Acad. **31** (1955), 141–146.

L'auteur définit une notion de distribution T multiple d'une distribution Q; il montre qu'elle revient à dire que T est limite de multiples usuels α_jQ, $\alpha_j\in\mathcal{E}$.

L. Schwartz.

Korevaar, Jacob. Distributions defined by fundamental sequences. III. Convergence. Convolution. Definite integral. Inverse Laplace transformation. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 483–493.

Continuation d'articles précédents [mêmes Proc. **58** (1955), 368–378, 379–389; MR **17**, 63]: propriétés essentielles des distributions définies comme limites de „suites fondamentales" de fonctions. Le titre indique les parties traitées. La transformation de Laplace est donnée pour des distributions à support la demi-droite $t\geqq 0$ et à croissance exponentielle. Pour qu'une fonction holomorphe $\Phi(s)$ soit image de Laplace d'une telle distribution, il faut et il suffit qu'elle soit holomorphe dans un demi-plan $\Re\, s>s_0$, et y soit majorée en module par un polynome en $|s|$. Formule d'inversion: si $\Phi(s)=\mathcal{L}T_t$, alors

$$T_t=\lim_{\sigma\to\infty,\delta\to\infty}(2i\pi)^{-1}\int_{\sigma-i\delta}^{\sigma+i\delta}\Phi(s)\exp(ts)ds\quad(\sigma>s_0)$$

avec convergence au sens des distributions.

L. Schwartz (Paris).

Korevaar, Jacob. Distributions defined by fundamental sequences. IV. Multiplication and division. Substitution. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 494–503.

Continuation d'articles précédents: propriétés essentielles des distributions définies comme limites de „suites fondamentales" de fonctions. Les parties traitées sont indiquées dans le titre. La substitution est le changement de variables: définition de $T_{g(t)}$, en particulier $\delta_{g(t)}$; formule $(T_{g(t)})'=T'_{g(t)}g'(t)$.

L. Schwartz (Paris).

See also: Newns, p. 362; Götc, p. 384.

Functions of a Complex Variable, Generalizations

Han Khwat Tik, and Kuipers, L. On a proof of the Lucas theorem concerning the zeros of the derivative of a polynomial. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 435–437.

The authors define $S(a,\theta,\varphi)$ as the sector $\theta\leqq\arg(z-a)\leqq\theta+\varphi$ and $S_2(a,\theta,\varphi)$ as the sum of $S(a,\theta,\varphi)$ and $S(a,\theta+\pi,\varphi)$. They then state that, if all the zeros of $f(z)=(z-z_1)\cdots(z-z_n)$ lie in $S(a,\theta,\varphi)$ and if $|\lambda|=1$ and $e^{i\varphi}\neq b/\bar{b}$, all the zeros of

$$F(z)=f(a+b(z-a))-\lambda f(a+e^{i\varphi}\bar{b}(z-a))$$

lie in $S_2(a,\theta-\frac{1}{2}a,\varphi)$. The proof follows from the fact that if z is not in this region,

$$|a+b(z-a)-z_k|-|a+e^{i\varphi}\bar{b}(z-a)-z_k|$$

has the same sign for all k. Lucas' Theorem is obtained as a limiting case.

M. Marden (Milwaukee, Wis.).

Fekete, M., and Walsh, J. L. On the asymptotic behavior of polynomials with extremal properties, and of their zeros. J. Analyse Math. **4** (1955), 49–87.

To every polynomial (1) $A_n(z)=z^n+a_1z^{n-1}+\cdots+a_n$, the authors associate a finite, non-negative number $N(A_n:E)$, its norm on the set E. N is called a quasi-

Tchebycheff (q.T.) norm if

(a) $$\sup_{a_i} N(A_n:E)/\max_{z\in E}|A_n(z)|<\infty,$$

(b) given an ε, the ratio in (a) has, for $n\geqq n_0(\varepsilon)$, a positive lower bound $L_n(E,\varepsilon)$ such that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} L_n(E,\varepsilon)^{1/n}=1.$$

The class of q.T. norms is very large. In the first place, it embraces the usual Tchebycheff norm as well as line and surface integral norms which themselves have been extended to the general form

$$\int_B w(z)h[|A_n(z)|]|dz| \text{ or } \iint_E w(z)h[|A_n(z)|]dS.$$

Here $w(z)$ is a non-negative continuous weight function, while $h(r)$ is a super-multiplicative Hausdorff function. Secondly, q.T. norms have a considerable algebra: If $N=N(A_n:E)$ is q.T., so is $C_nN(A_n:E)$ where C_n is a constant with $0<C_n\leqq C_0<\infty$, $\lim C_n^{1/n}=1$. If N_1 and N_2 are q.T., so is an arbitrary mean $N(N_1, N_2)$ which has the property that $\min(N_1, N_2)\leqq N(N_1, N_2)\leqq\max(N_1, N_2)$. Any finite linear combination of q.T. norms with positive coefficients is q.T.

The paper proceeds with an extensive study of polynomials which are minimal with respect to a q.T. norm over an infinite, closed, bounded point set E. The authors obtain theorems which give for these minimal polynomials their asymptotic character, their zero-distribution, their relation to $\tau(E)$ the transfinite diameter of E, and to the mapping function of the complement of E. Some of these theorems generalize well-known results which had been previously proven for special norms.

The following three extremal polynomials are studied: A_n^* is a polynomial (1) which minimizes a given q.T. norm N. A_n^{**} minimizes N among the polynomials (1) whose zeros belong to E. A_n^{***} minimizes N among the polynomials (1) whose zeros lie on B the boundary of E. Assume A_n^* exists for all n, and set

$$\nu_n=N[A_n^*],\quad \mu_n=\max_{z\in E}|A_n^*(z)|,$$

$$\lambda_n=N[A_n^{**}],\quad \varrho_n=\max_{z\in E}|A_n^{**}(z)|.$$

Then $\lim \nu_n^{1/n}=\lim \mu_n^{1/n}=\lim \lambda_n^{1/n}=\lim \varrho_n^{1/n}=\tau(E)$. Similar results for A_n^{***}.

Special attention has been paid to cases where it may be convenient to define a norm on E as the limit of norms on monotonic increasing subsets of E. The relation between the generalized Green's function and the transfinite diameter is then studied.

Let $\varphi(z)=\exp[G(z)+iH(z)+\log\tau(E)]$, where $G(z)$ is the generalized or ordinary Green's function with pole at ∞ of comp (E), $H(z)=\text{conj}\,(G)$. Then, for any q.T. norm N, we have on comp (E),

$$\lim|A_n^{**}(z)|^{1/n}=\lim|A_n^{***}(z)|^{1/n}=|\varphi(z)|.$$

The same is true of $A_n^*(z)$ provided E is convex and N is monotonic in the sense of underpolynomials.

The asymptotic distribution of the zeros of the polynomials A_n^* arising from a monotonic q.T. norm is related to certain generalized harmonic measures. Special results are obtained on the location of the zeros of A_n^* when the norm is $\int_B w(z)|A_n(z)|^2|dz|$ and when E and w are symmetric in the origin. *P. Davis.*

Fekete, M., and Szegö, G. On algebraic equations with integral coefficients whose roots belong to a given point set. Math. Z. **63** (1955), 158–172.

In this paper P denotes the field of rational numbers and K the class of equations

$$E(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n=0$$

which have rational integral coefficients and which are irreducible in P. Similarly, $P(\theta)$ denotes the algebraic field generated by a root θ of an equation $z^2+pz+q=0$ with rational integral coefficients and negative discriminant, and $K(\theta)$ denotes the class of equations $E(z)=0$ which have coefficients that are integers in $P(\theta)$ and which are irreducible in $P(\theta)$. Fekete [Math. Z. **17** (1923), 228–249] had proved that there are only a finite number of equations which belong to K and which have all their roots in a given bounded closed set S with a transfinite diameter $d<1$. In the present paper, the same theorem is established for the equations belonging to $K(\theta)$ and shown to be not valid for any other algebraic field than P or $P(\theta)$. The problem is studied in detail in the case that S contains a leminiscate with an equation of the form $|E(z)|=1$ and then the method of approximation by suitable lemniscate domains is used to investigate the more general regions S. *M. Marden* (Milwaukee, Wis.).

Hvedelidze, B. V. On a discontinuous problem of Riemann-Privalov in the theory of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1081–1084. (Russian)

The well-known Riemann boundary-value problem for analytic functions is studied, when the curves C involved constitute a finite collection of nonintersecting open arcs (the angle made by the tangent being of class (H)) and when the boundary values of the analytic solutions, that one seeks, may have an infinity of discontinuities. The original formulation of this problem (in the discontinuous case), as one in the theory of analytic functions, is due to Privalov [Mat. Sb. **41** (1934), 519–526]. Use is made of Lebesgue integration. Amongst the difficulties are those due to the presence of end points of the arcs of C.

W. J. Trjitzinsky (Urbana, Ill.).

Hvedelidze, B. V. On the Riemann-Privalov problem in the theory of analytic functions. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 165–171. (Russian)

C is a collection of closed or open, mutually nonintersecting smooth curves in the plane ω; $D=\omega-C$. The problem is to find $\varphi(z)$, analytic in D, of finite order at ∞, such that (1) $\varphi^+(t)=a(t)\varphi^-(t)+b(t)$ on C (Riemann, Plemelj, Carleman and others). Privalov's [Mat. Sb. **41** (1934), 519–526] problem is solution of (1), with $\varphi(z)$ representable by integrals of Cauchy type. The author solves Privalov's problem in the general formulation, when it is only required that $\varphi(z)$ be representable by integrals of Cauchy type. If $a(t)$, $b(t)$ satisfy Hölder conditions and $a(t)\neq 0$ on C, then the solution of (1), given by Cauchy integrals, is piece-wise analytic.

W. J. Trjitzinsky (Urbana, Ill.).

Iacob, Caius. Généralisation d'un théorème de Priwaloff. Com. Acad. R. P. Române **1** (1951), 433–437. (Romanian. Russian and French summaries)

Soit $\varphi(\theta)$ une fonction périodique, continue, de période

2π; on en définit la conjugée $\psi(\theta)$ par la formule

$$(1)\quad \psi(\theta)=\frac{1}{2\pi}\int_0^{2\pi}[\varphi(x)-\varphi(\theta)]\cot g\frac{\theta-x}{2}dx+\text{const.}$$

de P. Fatou. Supposons que $\varphi(\theta)$ admette un module de continuité $f(r)$ vérifiant les conditions suivants: 1) $f(r)$ est une fonction positive, non décroissante, continue et nulle pour $r=0$, telle que l'intégrale

$$(2)\quad g(r)=\int_0^r\frac{f(r)}{r}\,dr,\quad 0<r\leqq 2\pi,$$

existe et définisse une fonction $g(r)$ non décroissante, continue et nulle pour $r=0$; 2) on a l'inégalité

$$(3)\quad \int_r^a\frac{f(r)}{r^2}dr\leqq K(a)\frac{h(r)}{r},\quad 0<r\leqq a,$$

où $K(a)$ est borné supérieurement par un nombre fixe K et $h(r)$ est une fonction positive, non décroissante, continue et nulle pour $r=0$. On prouve alors que $\psi(\theta)$ admet un module de continuité de la forme $Ag(4r)+Bh(4r)$. Ce résultat généralise un théorème de J. Priwaloff [Bull. Soc. Math. France **44** (1916), 100–103]. (French summary.) The theorem in question, usually attributed to Priwaloff, was published at an earlier date by A. Korn [Abh. Preuss. Akad. Wiss. **1909**, Anhang, Abh. II]. *R. Finn.*

Kuribayasi, Akikazu. On functions of bounded Dirichlet integral. Kōdai Math. Sem. Rep. **7** (1955), 30–32.

Let B be a schlicht domain in the z-plane which contains the origin $z=0$ and whose boundary Γ consists of n analytic curves. Let χ be the class of all analytic functions in B which vanish at the origin. The function pair $\varphi_m(z)$ and $\phi_m(z)$ is determined in a unique way by the conditions: (a) $\varphi_m(z)$ and $\phi_m(z)-z^{-m}$ lie in χ; (b) $d\varphi_m=d\bar{\phi}_m$ on Γ. Let $f(z)\in\chi$ have the development $\sum c_\mu z^\mu$ at the origin and let $\varphi_m(z)=\sum S_{m\mu}z^\mu$ there. In order that $f(z)$ have a Dirichlet integral $D\{f\}=\int_B|f'|^2\leqq\pi$ it is necessary and sufficient that all inequalities

$$\left|\sum_{\mu=1}^N\mu c_\mu x_\mu\right|^2\leqq\sum_{\mu,\nu=1}^N\nu S_{\mu\nu}x_\nu\bar{x}_\mu\quad(N=1,2,\cdots)$$

be fulfilled. The necessity is an immediate consequence of Grunsky's boundary-integration technique and of the definition of the $\varphi_m(z)$. In order to show the sufficiency, the coefficients c_ν are used at first to define a linear operator in the function spaces $\sum_{\nu=1}^N\bar{x}_\nu\varphi_\nu'(z)$. By the Hahn-Banach theorem the operator is extended into the full Hilbert space of analytic functions in B with finite Dirichlet integral, and is identified with an analytic function $f(z)$ with $D\{f\}\leqq\pi$. An extension to the case that B lies on a Riemann surface is given and an analogous condition is obtained. *M. Schiffer* (Stanford, Calif.).

Rubel, L. A. Necessary and sufficient conditions for Carlson's theorem on entire functions. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 601–603.

Let $f(z)$ belong to the class K of entire functions of exponential type which obey the restriction $f(iy)=O(1)\exp c|y|$ for some $c<\pi$. A set A of positive integers is said to be a C-set if $f(n)=0$ for all $n\in A$ implies that $f(z)\equiv 0$. If A is a C-set, it is necessary that $\bar{D}(A)=0$ where $\bar{D}(A)=\limsup A(n)/n$, the upper density of A. The author outlines a proof that this condition is also sufficient. This answers for the class K the same question that W. H. J. Fuchs [J. London Math. Soc. **21** (1946), 106–110; MR **8**, 371] answered for the more extensive class K^* of functions of exponential type in the right half-plane. It is interesting to note that the criterion is not the same. A set A may be a C-set for K and not for K^*. *R. C. Buck.*

Buck, R. C. On n-point expansions of entire functions. Proc. Amer. Math. Soc. **6** (1955), 793–796.

The methods of an earlier paper [Trans. Amer. Math. Soc. **64** (1948), 283–298; MR **10**, 693] are extended to cover expansions of entire functions of arbitrary exponential type. The details are given for a generalised Lidstone series when the function is determined by the expansion apart from a finite trigonometric sum.

A. J. Macintyre (Aberdeen).

Strelic, Š. On maximum moduli of analytic functions. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 153–160. (Russian)

Two functions, $f(z)$, $g(z)$, are called essentially different if neither of the relations $f(z)=\varepsilon g(\eta z)$, $f(z)=\varepsilon\overline{g(\eta\bar{z})}$ are satisfied identically in z, for any ε and η, $|\varepsilon|=|\eta|=1$. Theorem: Suppose $M(z)=a_0+a_1z+a_2z^2+\cdots$ $(a_0\neq 0)$ is analytic for $|z|<1$, with real coefficients, and (a) $M(r)>0$, $r>0$, (b) $rM'(r)/M(r)>0$, $r>0$, (c) $K(z)=zM'(z)/M(z)$ is a bounded analytic function for $|z|<R$ $(R>1)$, assuming a positive value only once on $|z|=r$. Then there exist an infinite number of essentially different functions $f(z)$, regular for $|z|<1$, whose maximum modulus is $M(r)$. ($M(z)=e^z$ is an example of a function satisfying the hypotheses.) [Reviewer's note: An example of two essentially different analytic functions whose $M(r)$ functions agree in an annulus was given by W. H. J. Fuchs, Proc. London Math. Soc. (2) **48** (1943), 35–47; MR **5**, 36.]

The author gives an example of an entire function $f(z)$, such that the left and right derivatives of the maximum-modulus function $M(r)$ differ at a point. *E. Reich.*

Bazilevič, I. E. On a case of integrability in quadratures of the Loewner-Kufarev equation. Mat. Sb. N.S. **37(79)** (1955), 471–476. (Russian)

The equation in question is

$$(1)\quad \frac{\partial f(z,t)}{\partial t}=-f(z,t)p(f,t),$$

where for $|u|<1$, $0\leqq t<\infty$, $\Re p(u,t)>0$, $p(u,t)$ is regular in u and piecewise continuous in t. When $p(f,t)$ has a special form, (1) can be transformed into a Bernoulli-type equation and integrated. As a result the author proves that the principal branch of

$$(2)\quad w(z)=\left[\frac{m}{1+a^2}\int_0^s(p(s)-ai)s^{-mai/(1+a^2)-1}f_*^{m/(1+a^2)}(s)ds\right]^{(1+ai)/m}$$

is univalent and regular in $|z|<1$, under the following conditions: $f_*(z)$ and $p(z)$ are regular, $f_*(z)$ is univalent starlike, and $\Re p(z)>0$ in $|z|<1$; and a and $m>0$ are real numbers.

The class of univalent functions of the form (2) contains as a proper subclass the starlike and spiral functions. For the class defined by (2), it is proved that $|a_n|\leqq n$.

A. W. Goodman (Lexington, Ky.).

Lebedev, N. A. On parametric representation of functions regular and univalent in a ring. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 767–768. (Russian)

The author generalizes the classical Loewner differ-

ential equation [Math. Ann. **89** (1923), 103–121] to the case of a function $f(z)$ mapping the ring domain R, $m<|z|<M$, conformally onto the w-plane cut by two Jordan curves ending respectively at $w=0$ and $w=\infty$, with $f(1)=1$. His result is somewhat more general than that obtained earlier for the same problem by Li [same Dokl. (N.S.) **92** (1953), 475–477; MR **15**, 516].

A. W. Goodman (Lexington, Ky.).

Maksimov, Yu. D. On locally ε-convex and locally ε-starlike multivalent functions. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 965–967. (Russian)

The author continues his work on the classes $C_\varepsilon(p, q)$ and $S_\varepsilon(p, q)$ defined earlier [same Dokl. (N.S.) **100** (1955), 1041–1044; MR **16**, 810]. He now announces bounds for the radius of convexity for $f(z) \in C_\varepsilon(p, q)$, bounds for the radius of starlikeness for $f(z) \in S_\varepsilon(p, q)$, and upper and lower bounds for $|f(z)|$ for $f(z) \in C_\varepsilon(p, q)$.

A. W. Goodman (Lexington, Ky.).

Mügel, Karl Wilhelm. Über meromorphe periodische Funktionen. Math. Nachr. **13** (1955), 187–230.

A meromorphic function $f(z)$ with the period $2\pi i$ is a meromorphic function $\varphi(w)$ of $w=e^z$ except for the values $w=0$ and $w=\infty$. This paper obtains directly the properties of periodic functions which follow from this remark, but in such detail that it should be noted also as a contribution to the theory of functions with two isolated essential singularities. *A. J. Macintyre* (Aberdeen).

Oguztörel, Namik. Extension de la théorie de Nevanlinna aux domaines multiplement connexes. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **18** (1953), 384–419.

Abweichend von früheren Untersuchungen über meromorphe Funktionen mit mehrfachzusammenhängenden Existenzbereichen [vgl. insbes. af Hällström, Acta Acad. Abo. **12** (1940), no. 8; MR **2**, 275] in denen von vornherein eine bestimmte Ausschöpfung des Existenzbereiches durch die Niveaulinien einer Potentialfunktion zugrunde gelegt wurde, führt Verf. die Wertverteilungsgrößen ganz allgemein in Abhängigkeit von einem Gebiet D ein, das einschl. seines Randes C ganz im Innern des Existenzgebietes liegt und sonst nur geeignete Regularitätsbedingungen erfüllt. Als Ausgangspunkt für die Gewinnung der beiden Hauptsätze dient die für das Gebiet D angesetzte Jensen-Nevanlinnasche Formel. Dadurch liegt die folgende Definition der Wertverteilungsgrößen nahe:

$$M(f, z, D)=\frac{1}{2\pi}\int_C \log|f(\zeta)|\frac{\partial G(\zeta, z, D)}{\partial n_\zeta}ds_\zeta,$$

$$N(f, z, D)=\sum_{\kappa=1}^{k} G(z, b_\kappa, D)$$

[$b_1, \cdots, b_k$; Pole von $f(z)$ in D; $G(z, b_\kappa, D)$: Greensche Funktion], und es ist auf diese Weise möglich, die klassische Theorie ohne größere Änderungen zu übertragen. So ergeben sich ganz analog: der erste Hauptsatz, die wesentlichen Eigenschaften der Charakteristik, z.B. deren Unbeschränktheit im Falle verschwindender Randkapazität des Existenzgebietes und ihre Darstellbarkeit durch eine sphärische Normalform, ferner die Formeln von Cartan und Frostman. Erst zum Beweis des 2. Hauptsatzes wird D durch die von den Niveaulinien der Greenschen Funktion berandeten „Greenschen Gebiete" ausgeschöpft. Abweichend vom 1. Hauptsatz tritt im 2. ein wesentlich von D abhängiges Zusatzglied auf (vgl. hierzu auch obige Arbeit von Hällström). Ist die Charakteristik beschränkt, so ist wie im klassischen Fall eine Quotientendarstellung durch beschränkte Funktionen möglich, ist sie Null, so ist $f(z)$ beschränkt und durch ein verallgemeinertes Blaschke-Produkt darstellbar. Für unbeschränkte Charakteristik gelten analoge Defektrelationen wie im klassischen Fall. Hierbei wird offenbar die Existenz einer Greenschen Funktion angenommen (positive Randkapazität). Schließlich wird auf die Übertragbarkeit weiterer Sätze wie Picard-Borel, Ahlfors-Frostmann hingewiesen und eine sich leicht aus der Jensen-Nevanlinnaschen Formel ergebende allgemeine Fassung des Phragmén-Lindelöfschen Satzes angegeben.

P. Seibert (Zbl **52**, 302).

Ohtsuka, Makoto. On exceptional values of a meromorphic function. Nagoya Math. J. **9** (1955), 119–121.

Let $f(z)$ be meromorphic in an open set D, and let E be a subset of C of zero inner harmonic measure with respect to D. The author proves that if $z_0 \in E$ and if $z_0 \in \overline{C-E}$, then $f(z)$ assumes every value in $S_{z_0}^{(D)}-S_{z_0}^{(C-E)}$ except for a set of outer logarithmic capacity zero, in each neighborhood of z_0 [for notation see Noshiro, Jap. J. Math. **17** (1940), 37–96; MR **2**, 81]. This extends certain results due to Brelot, Noshiro and Tsuji.

M. O. Reade (Ann Arbor, Mich.).

Warschawski, S. E. On a theorem of L. Lichtenstein. Pacific J. Math. **5** (1955), 835-839.

The object of this note is the proof of the following theorem. Let C be a closed Jordan curve in the z-plane which possesses a corner of opening $\pi\alpha$, $0<\alpha\leqq 2$, at $z=0$. Suppose that this corner is formed by two regular analytic arcs

$$z=\sum_{\nu=1}^{\infty} a_\nu t^\nu, \quad z=\sum_{\nu=1}^{\infty} b_\nu t^\nu$$

where $0\leqq t\leqq 1$, $a_1\neq 0$, $b_1\neq 0$. If $\zeta=f(z)$ maps the interior Δ of C conformally on the half-plane $\Im[\zeta]>0$ so that $f(0)=0$, then for every integer n

$$\lim_{z\to 0}\left\{z^{n-1/\alpha}\frac{d^n f}{dz^n}\right\}=c\frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)\cdots\left(\frac{1}{\alpha}-n+1\right),$$

for unrestricted approach, where $\lim_{z\to 0}\{z^{-1/\alpha}f\}=c$.

This result was stated by L. Lichtenstein [J. Reine Angew. Math. **140** (1911), 100–119], but proved by him only for irrational values of α. *E. T. Copson.*

Hersch, Joseph. Contribution à la théorie des fonctions pseudo-analytiques. Comment. Math. Helv. **30** (1956), 1–19 (1955).

Soit $R^{(z)}$ une surface de Riemann: R son support topologique, (z) sa structure conforme. Soit $G^{(z)}$ (de support G) $\subset R^{(z)}$ et $p \in G^{(z)}$. Une transformation intérieure $w(p)$ de $G^{(z)}$ dans S est D-pseudo-analytique si tout quadrilatère Q contenu dans un voisinage suffisamment petit $U(p)$ est transformé (eventuellement sur une surface de recouvrement ramifiée de S) en un quadrilatère Q' avec la relation $D^{-1}\mu\leqq\mu'\leqq D\mu$, où D est une constante fixe et μ et μ' les modules des quadrilatères Q et Q'; si de plus $w(p)$ est topologique, elle est D-pseudo-conforme. Il est alors possible d'introduire sur R une structure conforme (w) par inversion locale de $w(p)$ telle que toute fonction D-pseudo-analytique puisse être mise sous la forme $w=P\circ F$, où F est D-pseudo-conforme ($F: G^{(z)}\to G^{(w)}$ ayant même support topologique) et P analytique.

L'auteur étudie alors la variation de certains invariants

conformes, par les représentations pseudo-analytiques, après avoir remarqué que l'inégalité sur les modules des quadrilatères Q et Q' s'étend globalement; on a des inégalités analogues pour les modules de domaines doublement connexes. La fonction $\nu(r)$ définie comme module du domaine doublement connexe dont les contours sont $|z|=1$ et le segment réel $0\leq x\leq r$ permet alors de donner de nouvelles inégalités sur la variation de la mesure harmonique et de la distance hyperbolique. D'intéressantes généralisations du théorème de Schottky, de l'inégalité de Jensen, du théorème de Phragmén-Lindelöf sont obtenus en complétant le schéma $w=P\circ F$ par une application T conforme de $F(G)$ sur $G^{(2)}$ de sorte que $T\circ F$ est une application D-pseudo-conforme de G sur lui même et $P\circ T^{-1}$ une fonction analytique à laquelle on peut appliquer par example le théorème de Schottky ou le théorème de Blaschke.

L'auteur étudie la variation de la longueur extrémale (Ahlfors-Beurling) par les transformations D-quasi-conformes qui sont les transformations intérieures à dérivées partielles continues pour lesquelles le quotient des dilatations reste $\leq D$. *L. Fourès* (Marseille).

See also: Dufresnoy et Pisot, p. 349; Sagawa, p. 358; Bourion, p. 361; Koizumi, p. 361; Sunouchi, p. 361; Schwarz, p. 370.

Harmonic Functions, Potential Theory

Demidovici, B. P. On a simple demonstration of the theorem of the mean for harmonic functions. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **9** (1955), no. 2, 58–59. (Romanian)

Translation of Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3 (61), 213–214; MR **16**, 129.

Dinghas, Alexander. Konvexitätseigenschaften von Mittelwerten harmonischer und verwandter Funktionen. Math. Z. **63** (1955), 109–132.

Let $(x_1, x_2, \cdots, x_n)$ be a point in the euclidean space R^n, $r=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$, and $\cos\vartheta=x_1/r$. L_0 denotes the class of functions $u(P)$ subharmonic in the half space H^n $(x_1>0)$, with continuous derivatives up to the second order, and satisfying at the finite frontier points Q of H the condition $\limsup u(P)\leq 0 (P\in H^n, P\to Q)$. L_k denotes the class of subharmonic functions that are limits of decreasing sequences of elements of L_0. With $u\in L_k$, the mean value $m_k(r)$ is defined by $m_k(r)=\int_{S_r}\psi^{2k}(P)\cos^2\vartheta\, d\omega_n$ where S_r is the intersection of H^n with the sphere $x_1^2+\cdots+x_n^2=r^2$, $d\omega_n$ is the surface element on S_1, and $\psi(P)=u^+(P)/\cos\vartheta$. It is first proved that for all $k\geq 1$ $(m_k(r))^{1/k}r^{n-2}$ is a convex function of r^n and $(m_k(r))^{\frac{1}{2}k}r^{-1}$ is a convex function of r^{-n}, results which were previously known only for $k=1$ and $k=\frac{1}{2}$. The class L_0 is generalized by replacing the inequality $\partial^2u/\partial x_1^2+\cdots+\partial^2u/\partial x_n^2\geq 0$ in the definition of $u\in L_0$ by the inequality $\partial^2u/\partial x_1^2+\cdots+\partial^2u/\partial x_n^2\geq cur^{-2}$, where $c(P)$ is continuous in H^n and satisfies $c(P)+n-1\geq 0$. For such functions a modified form of the Phragmen-Lindelöf principle is obtained with a smaller gap in the possible values of

$$\liminf_{r\to\infty}\log M(r)/\log r,$$

where $M(r)$ is the maximum of $u^+(P)$ on S_r. Further theorems generalize the Hadamard three-circles theorem and related convexity theorems for functions subharmonic in R^n. *F. F. Bonsall* (Newcastle-on-Tyne).

Gabriel, R. M. An extended principle of the maximum for harmonic functions in 3-dimensions. J. London Math. Soc. **30** (1955), 388–401.

L'A. considère dans R^3 trois fonctions harmoniques $U(P)$, $V(Q)$, $W(R)$ de gradients non nuls au voisinage de P_0, Q_0, R_0, de telle sorte que P, Q, R variables restent alignés, R divisant le segment PQ dans un rapport b/a fixe, de sorte aussi que $U(P)=V(Q)$ et que $U(P)+V(Q)-2W(R)$ ait un maximum pour P_0, Q_0, R_0. Alors, par des considérations élémentaires mais assez longues l'A. montre que $U(P_1)=V(Q_1)=W(R_1)+\text{const.}$, où $\overrightarrow{P_0P_1}=\mathbf{r}/A$, $\overrightarrow{Q_0Q_1}=\mathbf{r}/B$, $\overrightarrow{R_0R_1}=\mathbf{r}/C$, avec A, B, C constantes et $\mathbf{r}$ vecteur variable (x, y, z) assez petit. Lorsque l'hypothèse du maximum est vraie quelque soit b/a, U, V, W sont constantes sur les rayons issus d'un certain point du segment P_0Q_0. Des applications sont faites à des domaines étoilés et aux surfaces de niveau de fonctions harmoniques. Soulignons que, dans un domaine convexe, les surfaces de niveau de la fonction de Green sont convexes. *M. Brelot* (Paris).

Sagawa, Akira. On the existence of Green functions. Tôhoku Math. J. (2) **7** (1955), 136–139.

Proof of the existence of the Green's function for a Riemann surface whose boundary consists of a finite number of contours. *M. Heins* (Providence, R.I.).

Kronberg, V. A. On the first variation of the solution of boundary problems of potential theory for variation of the bounding surface. Prikl. Mat. Meh. **19** (1955), 463–470. (Russian)

Verf. betrachtet die Änderung, die die Lösung $\varphi(p)$ einer der klassischen potentialtheoretischen Randwertaufgaben bei kleiner Änderung der Randfläche S und entsprechender Abänderung der Randwerte erfährt. Ist $\mathbf{r}(u, v)$ der Radiusvektor eines Punktes P von S (u, v krummlinige Koordinaten von P), so soll die neue Randfläche S^* durch $\mathbf{r}^*(u, v, \varepsilon)=\mathbf{r}(u, v)+\varepsilon\eta(u, v)\cdot\mathbf{n}$ charakterisiert sein ($\mathbf{n}$ Einheitsvektor der Außennormale zu S, ε kleiner Parameter). Man denkt sich ferner die Randfunktion in eine Umgebung von S, die S^* enthält, fortgesetzt und verwendet die Werte auf S^* als die Randwerte der neuen Aufgabe. Setzt man die Lösung dieser Aufgabe in der Gestalt

$$\varphi^*(p, \varepsilon)=\varphi(p)+\varepsilon\varphi^{(1)}(p)+\varepsilon^2\varphi^{(2)}(p)+\cdots$$

an, so wird gezeigt, daß $\varphi^{(1)}(p)$ als ein Integralausdruck dargestellt werden kann, der in $\eta(u, v)$ linear ist und von der Randfunktion der ursprünglichen Aufgabe abhängt. *K. Maruhn* (Dresden).

Pacelli, Mauro. Esame di una successione di potenziali di strato ellittico con applicazione a problemi armonici nello spazio e nel semispazio. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 1–22 (1 plate).

The author considers the Newtonian potential $V_s=\int_\sigma(\mu^s/r)d\sigma$ of an ellipse $x^2/a^2+y^2/b^2-1=0$, with density $\mu^s=(1-x^2/a^2-y^2/b^2)^{s/2}$, for $s=-1, 1, 3, 5, \cdots$. Here $\overline{V}_s$, the value of V_s in the interior points of the ellipse, is represented by a polynomial of degree $(s+1)/2$ and the coefficients are calculated in terms of elliptic integrals. It is proved that the $r+1$ partial derivatives

$$\partial^r\overline{V}_{2r-1}/\partial x^t\partial y^{r-t}\quad (t=0, 1, \cdots, r)$$

are linearly independent. Further, the so-called direct

potentials $D_s=\int_\sigma \mu^s r d\sigma$ are considered and it is proved that $\Delta\Delta D_s=0$.

Using the series of potentials V_s, the following problems are solved. 1. To determine a function φ regular and harmonic for $z>0$, so that $\lim_{R\to\infty} R\varphi$ and $\lim_{R\to\infty} R^2\partial\varphi/\partial R$ exist and so that φ satisfies the following conditions for $z=0$:

a) $\varphi=P_n(x, y)$ in the interior of the ellipse,
$\partial\varphi/\partial z=0$ in the exterior;

b) $\varphi=0$ in the interior,
$\partial\varphi/\partial z=Q_n(x, y)$ in the exterior,

P_n and Q_n being given polynomials. 2. To determine a function φ regular and harmonic except in the points interior to the ellipse, so that $\lim_{R\to\infty} R\varphi$ and $\lim_{R\to\infty} R^2\partial\varphi/\partial z$ exist, and so that for $x^2/a^2+y^2/b^2<1$

$$\lim_{z\to 0^+}\varphi(x, y, z)=P_n(x, y),\quad \lim_{z\to 0^-}\varphi(x, y, z)=Q_n(x, y),$$

P_n and Q_n being again given polynomials.

H. Bremekamp (Delft).

See also: Iacob, p. 355; Synge, p. 373; Malgrange, p. 404.

Series, Summability

Jurkat, W., and Peyerimhoff, A. The consistency of Nörlund and Hausdorff methods (solution of a problem of E. Ullrich). Ann. of Math. (2) **62** (1955), 498–503.

The regular, real Nörlund method of summability N_p is obtained by allowing the $\{p_n\}_0^\infty$ used in the definition of the regular, positive Nörlund method [G. H. Hardy, Divergent series, Oxford, 1949, pp. 64–65; MR **11**, 25] to be real and adding the two conditions $P_n\neq 0$ and $\sum_0^n |p_k|=O(P_n)$. All such methods N_p are consistent: If $s_n\to s(N_p)$ and $s_n\to s'(N_p)$, then $s=s'$ [Hardy, loc. cit., Theorem 18 modified as in Lemma 1 of the paper being reviewed]. All regular Hausdorff schemes H_χ (with $\chi(t+0)=\chi(t)$ for $0\leq t<1$) are also consistent [Hardy, loc. cit., Chap. 11]. The generalized Abel method of summability A^* is defined as follows: $s_n\to s(A^*)$ if $\sigma(x)=(1-x)\sum s_n x^n$ has a positive radius of convergence and defines a function which is holomorphic for $0\leq x<1$, is meromorphic for $|x|<1$, and tends to s as $x\to 1-0$. It is (essentially) known that if $s_n\to s(N_p)$, then there is an ε, $0<\varepsilon\leq 1$, such that σ is holomorphic for $1-\varepsilon\leq x<1$ and otherwise satisfies the other properties of σ above [Hardy, loc. cit., Theorem 18] and that, if $s_n\to s(A^*)$, then the sequence of H_χ transforms of $\{s_n\}_0^\infty$ is summable (A^*) to s [O. Szász, Ann. Soc. Polon. Math. **25** (1952), 75–84; MR **15**, 26]. The authors prove that if $\sum a_n$ is summable H_χ, $0<\varrho=\inf\{a;\ \chi(t)=\chi(1) \text{ for } a\leq t\leq 1\}<1$, $\sigma(x)=\sum a_n x^n$ is holomorphic at $x=0$, and σ, defined by analytic extension, is meromorphic in the disk $|x(2-\varrho)-(1-\varrho)|<1$, then σ is in fact holomorphic throughout this disk. (The result is stated for $0<\varrho\leq 1$, but the proof given is valid only for $0<\varrho<1$.) They combine these three results to show that if a sequence is N_p and H_χ summable to s and s', respectively, then it is summable A^* to $s''=s=s'$. The authors indicate that their results are equally valid for N_p, H_χ, and A^* summability of functions.

A. E. Livingston (Seattle, Wash.).

Knopp, Konrad. Nörlund-Verfahren für Funktionen. II. Math. Z. **63** (1955), 39–52.

This paper is concerned with the Nörlund summability method applied to functions instead of to series. Some basic theorems have been given in a previous paper by the author and B. Vanderburg [Rend. Circ. Mat. Palermo (2) **4** (1955), 5–32; MR **17**, 147]. Particular attention is given in the present paper to the summability and the absolute summability of the resultant of two functions. Three theorems are given which relate the summability of the resultant to the summability of its components. The results are of a comprehensive character and, because of the generality of the Nörlund method of summability, contain several known results.

H. G. Eggleston (Cambridge, England).

Wilansky, Albert, and Zeller, Karl. Inverses of matrices and matrix-transformations. Proc. Amer. Math. Soc. **6** (1955), 414–420.

Let $A=(a_{nk})$ be a matrix of complex numbers. Let D be the set of sequences $x=\{x_n\}$ such that $y=Ax$ is defined, where y is the sequence $\{y_n\}$ with $y_n=\sum_k a_{nk}x_k$ $(n=1, 2, 3, \cdots)$. Let R be the set of all Ax for $x\in D$. To distinguish between inverse matrix and inverse transformation let Ax define a transformation $T(x)$ from D onto R. The authors investigate various questions concerning (a) the existence of right, left, and two-sided inverses, denoted by A', $'A$, A^{-1}; (b) the same for T, denoted similarly; (c) connections between (a) and (b). From the many results obtained we select the following to indicate the nature of the problems considered. Theorem 1: (i) A linear T' always exists; (ii) $'T$ exists (and is linear) if and only if T is 1-1; (iii) if $'T$ exists it is unique, and $'T=T'=T^{-1}$; (iv) T' is unique if and only if $'T$ exists.

J. D. Hill (East Lansing, Mich.).

Kuttner, B. The problem of 'translativity' for Hausdorff summability. Proc. London Math. Soc. (3) **6** (1956), 117–138.

Let H_α be a Hausdorff method of summation generated by a function of bounded variation $\alpha(t)$, $0\leq t\leq 1$. H_α is translative if the field of all H_α-summable sequences, and the H_α-limit defined on it, are invariant under both left and right translation of sequences. In terms of the Mellin transform, $T(z)=\int_0^1 t^z d\alpha(t)$, conditions are given for H_α to be translative. Using sequences $\{s_n\}$ such that $\Delta^r s_0=(z-r)^{-k}$ $(r=0, 1, \cdots)$ it is shown that if H_α is translative, then $K(z)=T(z)/T(z+1)$ is regular for $\Re z>0$. If $K(z)$ satisfies conditions such as $K(it)-a\in L^2$ for $|t|\geq A$ (where a and A are some constants), which allow one to apply the Wiener-Pitt and Rogosinski theories and cannot be reproduced in detail here, $K(z)$ is a Mellin transform, and then the regularity of $K(z)$ in $\Re z>0$ and continuity in $\Re z\geq 0$ are sufficient for the translativity of H_α.

G. G. Lorentz (Detroit, Mich.).

Shapiro, G. On a convergence problem. Proc. Amer. Math. Soc. **6** (1955), 910.

The author shows that a conjectured theorem of A. Shields [Bull. Amer. Math. Soc. **60** (1954), 589] is false.

Henstock, R. The efficiency of convergence factors for functions of a continuous real variable. J. London Math. Soc. **30** (1955), 273–286.

In the theory of the generalized limits

$$S(s)=\lim_{x\to\infty} S(s, x)=\lim_{x\to\infty}\lim_{y\to\infty}\int_0^y a(x, u)ds(u),$$

the function $s(w)$ is usually assumed to be of bounded variation, in the ordinary or generalized sense. The present paper discusses the case when $s(w)$ is a bounded Baire function in each finite interval. The difficulties in the definitions of the Stieltjes integrals involved, which arise in this case, are pointed out in the following theorem: There is a bounded Baire function $s(w)$ such that the Perron-Stieltjes integral (1) $\int f(w)ds(w)$ does not exist for each non-constant function $f(w)$. To surmount this difficulty the author defines the integral (1) by

$$\int_\alpha^\beta f(w)ds(w)=f(\beta)s(\beta)-f(\alpha)s(\alpha)-(PS)\int_\alpha^\beta s(w)df(w),$$

and proves that the last integral exists for all bounded Baire functions s, if and only if f is of bounded variation. Using this definition the author obtains the following necessary and sufficient conditions on $a(x, w)$ in order that $S(s)$ should exist equal to $\lim s(w)-s(0)$ $(w\to\infty)$, whenever the latter exists: (i) $a(x, w\pm)$ exists for each $x\geqq 0$ and tends to 1 as $x\to\infty$. (ii) $M(x)=V[a(x, w)]<\infty$ for each $x\geqq 0$ and $\limsup M(x)<\infty$, as $x\to\infty$. (iii) For each $A>0$, $\lim_{0<w<A}\sum|(a(x, w+)-a(x, w-)|=0$, as $x\to\infty$. (iv) For each bounded Borel set W,

$$\lim_{x\to\infty}(LS)\int_W da_x(x, w)=0.$$

M. Collar (Mendoza).

Rosser, J. Barkley. Explicit remainder terms for some asymptotic series. J. Rational Mech. Anal. **4** (1955), 595–626.

Under general conditions the integral $\int_a^b f(x)g(x)dx$ can be written by means of integration by parts as $u_0+\cdots+u_{m-1}+\int_a^b f_m(x)g_m(x)dx$, where the terms $u_0, u_1, \cdots$ can be calculated easily and where the last integral denotes the remainder term. The present paper treats the following cases; in the cases (1), (2), (3) and (4) we have $a=0$ and $b=\infty$ and $f(x)=f_0(x)$. (1) $g(x)=x^re^{-Mx}$; $f_m(x)=M^{-m-r-1}f^{(m)}(x/M)$; $g_0(x)=x^re^{-x}$; $g_{n+1}(x)=\int_x^\infty g_n(t)dt$; if $-1<r\leqq 0$, then $g_m(x)$ is $\leqq g_0(x)$ and $\leqq e^{-x}g_m(0)$. (2) $g(x)=x^re^{-Mx^2}$; $f_{n+1}(x)=x^{-r}d(f_n(x)-f_n(0)x^r)/dx$; $g_m(x)=M^{-m}g(x)$. (3) $g(x)=e^{-Mx^2}$; $f_m(x)=M^{-(m+1)/2}f^{(m)}(xM^{\frac{1}{2}})$; $g_0(x)=e^{-x^2}$; $g_{n+1}(x)=\int_x^\infty g_n(t)dt$; then $g_m(x)$ is $\leqq e^{-x^2}(2x)^{-m}$ and $\leqq e^{-x^2}g_m(0)$. (4) $g(x)=e^{-Mx^2}$;

$$f_{n+1}(x)=\frac{d}{dx}\frac{f_n(x)-f_n(0)}{2x};\quad g_m(x)=M^{-m}e^{-Mx^2}.$$

(5) $g(x)=H^M(x)$, where $H(x)\geqq 0$ and $H'(x)\neq 0$; $f(x)=f_0(x)H'(x)/H(x)$; $f_{n+1}(x)=-f_n'(x)H(x)/H'(x)$; $g_m(x)=M^{-m}H'(x)H^{M-1}(x)$. (6) $g(x)=H^M(x)$, where $H(x)\geqq 0$ and there is exactly one φ $(a\leqq\varphi\leqq b)$ with $H'(\varphi)=0$; $H''(\varphi)<0$; $f(x)=f_0(x)$; $g_m(x)=M^{-m}H^M(x)$; $f_{n+1}(x)$ is either

$$\frac{d}{dx}\Big(-\frac{H(x)}{H'(x)}f_n(x)-\frac{f_n(\varphi)}{2\eta}\Big(\frac{2H(\varphi)}{-H''(\varphi)}\Big)^{\frac{1}{2}}\Big),$$

where $\eta=(\log(H(\varphi)/H(x)))^{\frac{1}{2}}$, or

$$f_{n+1}(x)=\frac{d}{dx}\Big(\frac{f_n(x)-f_n(\varphi)}{-H'(x)}H(x)\Big).$$

Some of these results are applied to the incomplete gamma function $\Gamma(z, y)=\int_y^\infty t^{z-1}e^{-t}dt$. Integration by parts yields for this integral an expansion in which the remainder term is $(z-1)(z-2)\cdots(z-m-1)\Gamma(z-m-1, y)$. The author treats this remainder term with the formula

$$(*)\quad y^Ne^y\Gamma(-N, y)\sim\frac{1}{y+N}-\frac{y}{(y+N)^3}-\frac{y(N-2y)}{(y+N)^5}-\frac{y(N^2-8Ny+6y^2)}{(y+N)^7}-\cdots,$$

which for instance gives an approximation correct to 8 decimals for $y=N=5$. The author writes the remainder term in (*) after m terms in the form

$$(**)\qquad(\Gamma(N+1))^{-1}N^{N-m}\int_0^\infty h_m(x)x^Ne^{-N}dx,$$

where $h_m(x)$ depends on $\gamma=y/N$. He deduces several properties of this function $h_m(x)$, for instance: if γ and $\operatorname{Re}x$ are positive, then $|h_m(x)|\leqq m!(\gamma+\frac{1}{2}\pi)^{-m}(\operatorname{Re}x)^{-1}$; if N and y are positive, then the integral in (**) is in absolute value $\leqq\Gamma(N+1)m!N^{m-N-1}(y+\frac{1}{2}\pi N)^{-m}$. The incomplete gamma function $\int_0^y t^{z-1}e^tdt$, defined in the whole complex z-plane by analytic continuation, is treated in another way. This function possesses an expansion consisting of $m+2$ terms with a certain remainder term, in which each term has an elementary form, except the term $C(m+1-z)/\Gamma(1-z)$, where $C(w)$ is defined by

$$\pi\cot\pi w+\Gamma(w)\int_0^w t^{-w}e^tdt$$

and by analytic continuation. Putting

$$\alpha=2^{\frac{1}{2}}(x-1-\log x)^{\frac{1}{2}}\text{ and }x^{-1}(1-x)^{-1}\frac{dx}{d\alpha}=\sum_{j=0}^\infty A_j\alpha^{j-1}$$

(the author calculates $A_1=2/3$; $A_3=23/270$; $A_5, \cdots, A_{11}$) he finds for $C(w)$ the asymptotic expansion

$$\Big(\frac{2\pi}{w}\Big)^{\frac{1}{2}}\sum_{j=0}^{\infty}{}'\frac{(2j)!A_{2j+1}}{j!(2w)^j};$$

Stieltjes derived the special case of this for w a positive integer. The author of this paper finds $|A_j|\leqq(3.8)(4\pi)^{-j/2}$ $(j\geqq 0)$ and he obtains estimates for the error term. Finally he writes the usual asymptotic expansion for $(2\pi N)^{\frac{1}{2}}N^{-n}e^N\Gamma(N+1)$ in the form $\sum_{j=0}^\infty(2j)!D_{2j}/j!(2N)^j$, where the coefficients D_j are defined by $dx/d\alpha=\sum_{j=0}^\infty D_j\alpha^j$; he finds $|D_j|\leqq(2.4)(4\pi)^{-j/2}$. This asymptotic expansion yields the coefficients D_{2j} for small j. The author approximates D_{2j} for large j by means of the numbers D_{2k} with small k; thus one can give approximate values for the later coefficients of the asymptotic series for the gamma function. *J. G. van der Corput* (Berkeley, Calif.).

See also: Dobbie, p. 348; Koizumi, p. 361; Sunouchi, p. 361.

Interpolation, Approximation, Orthogonal Functions

Szőkefalvi-Nagy, Béla. Polynomes positifs. I, II. Mat. Lapok **3** (1952), 140–147; **4** (1953), 13–17. (Hungarian. Russian and French summaries)

The first paper presents the well-known proof of the following theorem attributed to F. Hausdorff [Math. Z. 9 (1921), 74–129, esp. pp. 98–99]. Let $P(x)$ be a polynomial which is positive in the closed interval $[0, 1]$. Then the following representation holds: $P(x)=\sum_{m=0}^s u_mx^m(1-x)^m$, where $s>0$ is sufficiently large and the u_m are positive. The second paper points out that this theorem was published earlier by S. Bernstein [Soobšč. Har'kov. Mat. Obšč. (2) **13** (1912), 49–194]. His proof is reproduced which is different from that of Hausdorff and does not make use of the decomposition of the polynomial in linear factors. The extension to polynomials of several variables is immediate. *G. Szegő* (Stanford, Calif.).

Kakehashi, Tetsujiro. Integrations on the circle of convergence and the divergence of interpolations. I. Proc. Japan Acad. **31** (1955), 329–333.

Let $z_i^{(k)}$ $(i \leqq k;\ k=1, 2, \cdots)$ lie in $|z| \leqq 1$, and assume that $(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})/z^n$ converges in $|z|>1$, uniformly in any closed finite set, to a function $\lambda(z)$ which is single-valued, non-vanishing, and analytic there. Let K_ϱ $(\varrho>0)$ designate the class of functions possessing the representation

$$f(z)=\sum_{n=0}^{\infty} c_n \varrho^{-n} z^n;\ c_n=(2\pi)^{-1}\int_0^{2\pi} e^{-in\theta}\,dF(\theta)\ (n=0, 1, \cdots),$$

$$\limsup_{n\to\infty} |c_n|>0,$$

where $F(\theta)$ is a complex-valued function of bounded variation normalized by $F(0)=0$, $F(\theta-0)=F(\theta)$. Theorem: Let $f(z) \in K_\varrho$ $(\varrho>1)$. If $P_n(z:f)$ designates the polynomial of degree n which interpolates to $f(z)$ at $z_i(n+1)$, $i=1, 2, \cdots, n+1$, the sequence P_n diverges at every point of $|z|>\varrho$. Also, $\limsup_{n\to\infty} |\varrho^n z^{-n} P_n(z:f)|>0$ for $|z|>\varrho>1$. *P. Davis* (Washington, D.C.).

Šaginyan, A. L. Some problems of the theory of best approximation in space. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **6** (1953), no. 3., 1–13. (Russian. Armenian summary)

Functions in three variables are approximated by harmonic polynomials. Theorems of Bernstein, de la Vallée Poussin, Jackson type are derived which relate the measure of best approximation to the modulus of continuity of various derivatives. *P. Davis.*

Bourion, Georges. Sur un mode d'approximation polynomiale d'une fonction holomorphe. Bull. Sci. Math. (2) **79** (1955), 69–72.

Let D be a fixed domain containing the unit disk $|z|<1$, and let the function f be holomorphic in D, with $f(z)=\sum a_n z^n$ in $|z|<1$. Then there exists a sequence of polynomials

$$(1)\qquad P_k(z)=\sum_{n=0}^{n_k} a_n z^n+\sum_{n=n_k+1}^{m_k} b_n z^n$$

for which $\lim n_k=\infty$ and which converges to f uniformly (sense of Montel) in D. The author shows that if f has such a representation which satisfies the additional condition $m_k/n_k \to 1$, then the Taylor series of f has lacunary structure; that is, $a_n=a'_n+a''_n$, where the sequence $\{a'_n\}$ possesses Ostrowski gaps and the series $\sum a_n'' z$ converges in a disk of radius greater than 1. For those cases where the Taylor series of f does not have lacunary structure, a lower bound greater than 1 for $\liminf m_k/n_k$ is determined by the domain D. *G. Piranian* (Ann Arbor, Mich.).

Feldmann, I. Über durch Sturm-Liouvillesche Differentialgleichungen charakterisierte orthogonale Polynomsysteme. Publ. Math. Debrecen **3** (1954), 297–304 (1955).

Let p_i be a polynomial of degree i, satisfying the differential equation

$$Qy''+xy'=\lambda_i y;\ Q=ax^2+bx+c,\ \lambda_i \geqq 0,\ i=0, 1, 2.$$

Also let $p_2=(x-\alpha)p_1-Ap_0$, $A>0$. Finally let the equation in question have the solutions $y=p_n$, p_n a polynomial of degree n, $\lambda=\lambda_n>0$, $n\geqq 3$. Then apart from linear transformations, p_n is the Jacobi, Laguerre, or Hermite polynomial according as the degree of Q is 2, 1, 0, respectively. A similar conclusion is drawn under the assumption that the function $Q^{-1}\exp\{\int Q^{-1}x\,dx\}$ is integrable and the zeros of $p_2(x)$ are real and distinct. *G. Szegő.*

See also: Danese, p. 363; Carlitz, p. 363; Zuhovickiĭ, p. 388; Stesin, p. 414.

Trigonometric Series and Integrals

★ **Zygmund, Antoni. Trigonometrical series.** Dover Publications, New York, 1955. vii+329 pp. $1.85.

Reprint by photo-offset of the original edition [Warszawa-Lwów, 1935].

Sokolov, I. G. Remainder term of the Fourier series of differentiable functions. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 23–26. (Russian)

Soit $W^{(r)}$ la classe des fonctions de période 2π, r fois dérivables $(r\geqq 1)$, satisfaisant $|f^{(r)}(x)|\leqq 1$, et soit $E_n(W^{(r)})=\sup|f-S_n|$ quand $f \in W^{(r)}$, S_n étant la $n$ième somme partielle de Fourier de f. Précisant une estimation de Kolmogoroff [Ann. of Math. (2) **36** (1935), 521–526], l'auteur montre que

$$E_n(W^{(r)})\leqq n^{-r}(4\pi^{-2}\log n+7)\ (n\geqq 2).$$

J. P. Kahane (Montpellier).

Koizumi, Simiyuki. On integral inequalities and certain applications to Fourier series. Tôhoku Math. J. (2) **7** (1955), 119–127.

Let $\varphi(z)$ be regular in the unit circle, of class H^r, $r>1$, and with boundary function $\varphi(e^{i\theta})$. If $P(\varrho, t)$ denotes the Poisson kernel, set

$$K_r(\varrho, \theta)=\int_0^{2\pi} |\varphi'(\varrho e^{i\theta+it})|^r P(\varrho, t)\,dt,$$

$$g_s(\theta)=\left\{\int_0^1 (1-\varrho)^{s-1} K_s(\varrho, \theta)\,d\varrho\right\}^{1/s}.$$

The author shows that for $s>2$

$$\left\{\int_0^{2\pi} g_s^r(\theta)\,d\theta\right\}^{1/r}\leqq A(r, s)\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}$$

while for $0<s<2$, if $\varphi(0)=0$, then

$$B(r, s)\left\{\int_0^{2\pi}\{|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}\leqq\int_0^{2\pi} g^r{}_s(\theta)\,d\theta\Big\}^{1/r}.$$

Application of the above inequalities are made to obtain results on the strong summability of power series on the circle of convergence. *P. Civin* (Eugene, Ore.).

Kanno, Kôsi. Cesàro summability of Fourier series. Tôhoku Math. J. (2) **7** (1955), 110–118.

Let $\varphi(t)$ be an even periodic function, and let $\varphi_\beta(t)$, $\beta>0$, be its βth integral. If $\varphi_\beta(t)=o(t^\gamma)$, $\gamma>\beta>0$, as $t\to 0$ and

$$\int_0^t |d(u^\Delta \varphi(u))|=O(t),\ 0<t<\eta \text{ with } \Delta\geqq\gamma/\beta,$$

then the Fourier series of $\varphi(t)$ is summable (C, α) to zero at $t=0$, for $\alpha=(\Delta\beta-\gamma)/(\Delta+\gamma-\beta-1)$. For $\Delta=\gamma/\beta$ this reduces to a convergence criterion given by Sunouchi [same J. (2) **5** (1954), 238–242; MR **16**, 240]. *P. Civin.*

Sunouchi, Gen-ichirô. On the summability of power series and Fourier series. Tôhoku Math. J. (2) **7** (1955), 96–109.

Let $f(z)$ be regular in the unit circle and of class H^p.

Let $\sigma_n^\alpha(x)$ denote the (C, α) mean of the Fourier series of the boundary function $f(e^{ix})$ of $f(z)$, and let $t_n^\alpha(x)=n\{\sigma_n^\alpha(x)-\sigma_{n-1}^\alpha(x)\}$. The following are examples of the various integral inequalities the author obtains relating the above quantities. If $0<p\leqq 2$, $\delta>0$, and $\alpha=(1+\delta)/p$, then

$$\int_{-\pi}^{\pi}\{\sum_{n=1}^{\infty}|t_n^\alpha(x)|^2/n\}^{\frac{1}{2}}dx\leqq A(p,\alpha)\int_{-\pi}^{\pi}|f(e^{ix})|^p dx.$$

If $1<p\leqq 2$, and $\alpha=1/p$, then

$$\int_{-\pi}^{\pi}\{\sum_{n=1}^{\infty}|t_n^\alpha(x)|^2/n[\log(n+1)]^{2/p}\}^{p/2}dx\leqq A(p)\int_{-\pi}^{\pi}|f(e^{ix})|^p dx.$$

P. Civin (Eugene, Ore.).

Newns, W. F. A Fourier integral theorem for functions of intervals. Proc. Amer. Math. Soc. **6** (1955), 816–822.

The Fourier transform of an interval function $u(I)$ is defined in terms of the Burkill integral [Proc. London Math. Soc. (2) **22** (1923), 275–310]. A function $u(I)$ defined on subintervals of $(0, \infty)$ is said to belong to the class B^2 if u is integrable on every finite range and the upper integral $\int_0^{*\infty}\{u(I)\}^2/|I|$ is finite. If a finite interval I is divided into a set of subintervals I_k and the sum $\sum f(t_k)u(I_k)$ converges to a limit as the length of the longest interval in the subdivision tends to 0, when t_k is any point in I the limit is written $\int_I f(t)u(I)$. Then the following analogue of the Plancherel theorem for Fourier transforms holds: if $u\in B^2$, the Fourier cosine transform $g(x)=$l.i.m. $(2/\pi)^{\frac{1}{2}}\int_0^a \cos xt\ u(I)$ exists and belongs to L^2; $\int_0^\infty\{g(x)\}^2dx\leqq\int_0^{*\infty}\{u(I)\}^2/I$. If h is the ordinary Fourier cosine transform of g, then $\int_0^a u(I)=\int_0^a h(t)dt$ and $h(x)=u'(x)$ p.p., where $u'(x)$ is the derivative of $u(I)$. The inequality in the theorem can fail to be an equality, but if $U(I_0)=\int_I u(I)$ then $\int_0^\infty\{g(x)\}^2dx=\int_0^\infty\{U(I)\}^2/|I|$. The proof includes discussion of special cases.

J. L. B. Cooper (Cardiff).

Magnus, Wilhelm. A Fourier theorem for matrices. Proc. Amer. Math. Soc. **6** (1955), 880–890.

This paper was reviewed in an earlier version [Div. Electromag. Res. Inst. Math. Sci. New York Univ., Res. Rep. No. **BR-8** (1954); MR **16**, 785].

Hartman, S. Almost periodic functions. Prace Mat. **1** (1955), 323–343. (Polish. Russian and English summaries)

Expository paper.

Integral Transforms, Operational Calculus

Churchill, R. V. Generalized finite Fourier cosine transforms. Michigan Math. J. **3** (1955–56), 85–94.

Let $\cos k_n x$ $(n=1, 2, \cdots)$ be the characteristic functions of the Sturm-Liouville problem $y''(x)+k^2y(x)=0$, $y'(0)=0$, $hy(1)+y'(1)=0$, $h\neq 0$, where k_n are the positive roots of the equation $h\cos k_n=k_n\sin k_n$, and where $F(x)$ denotes any bounded integrable function on the interval $(0, 1)$. The operational mathematics of the linear integral transformation,

$$T\{F(x)\}=\int_0^1 F(x)\cos k_n x\,dx=f(n)$$

is developed. An elementary convolution property of the transformation, that is, a formula for the inverse transforms, is presented. The transformation is a special case of the finite Fourier transformations treated by Roettinger, whose convolution property is written in terms of almost periodic extensions of the functions from finite interval to the infinite interval. The method presented is applicable to the more general finite Fourier transformations. *S. Kulik* (Columbia, S.C.).

Griffith, J. L. On the asymptotic behaviour of Hankel transforms. J. Proc. Roy. Soc. New South Wales **88** (1954), 71–76 (1955).

A generalization of the results of a paper by the same author [same J. **88** (1955), 61–65; MR **16**, 1102]. With $D\equiv(1/x)(d/dx)$, $g(\nu, x)=x^{-\nu}g(x)$ and

$$\bar{g}(u)=\int_0^\infty xJ_\nu(xu)g(x)dx,$$

it is shown that for $\frac{1}{2}<a<p+\nu+2$ and with certain restrictions on $g(x)$,

$$\lim_{u\to\infty}\{u^{p+2-a}\bar{g}(y)\}=\gamma(p,\nu,a)\lim_{x\to+\infty}\{(-1)^n x^{p+\nu+a}D^p g(\nu,x)\}.$$

This is generalized to include the case where $g(x)$ may have a finite number of finite discontinuities, and a further result gives the behaviour of $D^p\bar{g}(\nu, x)$ as $x\to+0$ in terms of that of $g(x)$ as $x\to\infty$. *F. Goodspeed.*

★ **Calderon, A. P. Singular integrals.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 319–328. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

The author summarizes the principal results obtained by Calderón and Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637] concerning the integral operators of the form

$$L_k f(x)=\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon}k(x-t)|x-t|^{-n}f(t)dt,$$

where x, t, denote points of the n-dimensional euclidean space, $x=(x_1, \cdots, x_n)$, $|x|=(\sum x_i^2)^{\frac{1}{2}}$, and $k(x)$ is a kernel satisfying $k(ax)=k(x)$ for $a>0$, and $J(k)=\int_{|x|=1}k(x)d\sigma=0$.

Some new and unpublished results are announced. For instance: for a fixed $q>1$, all the operators $\lambda I+L_k$, $k\in L^q$, form a semisimple Banach Algebra, the norm being defined by $\|\lambda I+L_k\|=|\lambda|+J(|k|^q)^{1/q}$. The maximal ideals of this algebra are the points x of the unit sphere $|x|=1$, and $(\lambda I+L_k)(x)=\lambda+\hat{k}(x)$ the Fourier transform of $k(x)|x|^{-n}$. *M. Cotlar* (Mendoza).

★ **Horváth, J. Hilbert transforms of distributions.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 61–70. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

Let (D_{L^p}) $(1\leqq p<\infty)$ be the space of functions φ such that φ and all its derivatives belong to $L^p(R^n)$, and let (D_{L^p}') be the dual of (D_{L^p}), $1/p+1/q=1$. For any function $k(\sigma)$ defined on the unit sphere S^{n-1} (and satisfying $\int_{S^{n-1}}k(\sigma)d\sigma=0$, $k(\sigma)\in L^s(S^{n-1})$, $s>1$), consider the distribution $K=$ v.p. $k(\sigma)/r^n$, $r=|x|$, $\sigma=x/r$, v.p.=principal value. For any distribution $T\in(D_{L^p}')$, the author defines the generalized Hilbert transform of T as $K(T)=T*K$, and extends this definition to vector-distributions of the form $\mathbf{T}=e_1T_1+\cdots+e_nT_n$, where $\{e_i\}$ is a canonical basis

of an Algebra and $\sum e_i^2=0$. Using the theory of distributions, the author deduces the Mihlin-Giraud composition formulas of singular integrals. *M. Cotlar.*

★**van der Pol, Balth., and Bremmer, H. Operational calculus based on the two-sided Laplace integral.** Cambridge, at the University Press, 1955. xiii+415 pp. $11.00.

The first edition [1950] was reviewed in MR **12**, 407. Although minor changes and corrections have been made, this edition is substantially like the first.

Mikusiński, J. Operational calculus in the light of present-day mathematical investigations. Prace Mat. **1** (1955), 344–370. (Polish. Russian and English summaries)

Expository paper.

See also: Korevaar, p. 354; Mura and Kinosita, p. 374.

Special Functions

Danese, Arthur E. Explicit evaluations of Turán expressions. Ann. Mat. Pura Appl. (4) **38** (1955), 339–348.

Let $P_n^{(\lambda)}(x)$ denote as usual the ultraspherical polynomials. The expression $[P_n^{(\lambda)}(x)]^2-P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x)$ is represented as a linear combination of the functions $[P_i^{(\lambda-1)}(x)]^2$, $0\leq i\leq n$. The coefficients are given in explicit form. This leads to a generalization of the "inequality of Turán" which has been considered recently by many authors. The analogous expression in which $P_n^{(\lambda)}(x)$ is replaced by the derivative of $P_{n+1}^{(\lambda)}(x)$ is dealt with in a similar fashion; here the basic polynomials are then $[P_i^{(\lambda)}(x)]^2$. Other identities of similar nature are obtained for the Laguerre and Hermite polynomials as well as for certain Bessel functions. *G. Szegő* (Stanford, Calif.).

Carlitz, L. A note on Hermite polynomials. Amer. Math. Monthly **62** (1955), 646–647.

The author proves the expansion of $H_m(x)H_n(x)$ as a sum of Hermite polynomials by induction, using the recurrence relation of Hermite polynomials, and thus extends the formula to any solution of the recurrence relations. *A. Erdélyi* (Pasadena, Calif.).

Gjellestad, Guro. Note on the definite integral over products of three Legendre functions. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 954–956.

The author evaluates $\int_{-1}^{1}[P_n^m(x)]^2P_k(x)dx$ for $k=2, 4$, and investigates those cases in which the integral vanishes. *A. Erdélyi* (Pasadena, Calif.).

Gran Olsson, R. An orthogonal property of the hypergeometric polynomial. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 59–61.

"This short note contains only remarks and corrections of some ambiguities and misprints in a paper of the same title written by H. Bateman" (Author's introduction). The paper in question appeared in Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 374–377 [MR **4**, 83]. *A. Erdélyi.*

Curtz, T. B., and Siegel, K. M. Inequalities involving cylindrical functions. Proc. Amer. Math. Soc. **6** (1955), 823–827.

Inequalities are derived relating $J_\nu(\nu x)$ and $J_\nu'(\nu x)$ to the same functions for the particular value $x=1$. Related results were previously given by Siegel [same Proc. **4** (1953), 858–859; MR **15**, 422] and Siegel and Sleator [ibid. **5** (1954), 337–344; MR **15**, 956]. A few similar results for Neumann's function are also given.

C. J. Bouwkamp (Eindhoven).

Guy, Jean, et Tillieu, Jacques. Calcul des intégrales de la forme

$$C_m=\int_{+1}^{\infty}\lambda^m(\lambda^2-1)^{\frac{1}{2}}e^{-a\lambda}d\lambda$$

(*m* entier). J. Phys. Radium (8) **16** (1955), 801–802.

The integral mentioned in the title is $(-d/da)^mK_1(a)/a$, and can be expressed as a finite combination of modified Bessel functions. The author gives these expressions for $m=0, 1, 2, 3, 4$. *A. Erdélyi* (Pasadena, Calif.).

Yih, Chia-Shun. Solutions of the hyper-Bessel equation. Quart. Appl. Math. **13** (1956), 462–463.

Solutions of the differential equation $L_p^n f=0$, $L_p=D^2+r^{-1}D-p^2r^{-2}+k^2$, are considered. If $p>0$ is not an integer, the functions $r^mJ_{\pm(p+m)}(kr)$, $m=0, 1, \cdots, n-1$, are solutions; otherwise the functions $r^mJ_{p+m}(kr)$ and $r^mN_{p+m}(kr)$, $m=0, 1, \cdots, n-1$, are solutions.

N. D. Kazarinoff (Lafayette, Ind.).

de Hoop, A. T. On integrals occurring in the variational formulation of diffraction problems. Nederl. Akad. Wetensch. Proc. Ser. B. **58** (1955), 325–330.

The author derives expressions for the integrals

$$\int_\alpha^\infty u^{-2}(u^2-\alpha^2)^{\frac{1}{2}}J_{2\mu}(u)J_{2\nu}(u)du,$$

$$\int_0^\alpha u^{-2}(\alpha^2-u^2)^{\frac{1}{2}}J_{2\mu}(u)J_{2\nu}(u)du,$$

$$\int_\alpha^\infty (u^2-\alpha^2)^{-\frac{1}{2}}J_{2\mu}(u)J_{2\nu}(u)du,$$

$$\int_0^\alpha (\alpha^2-u^2)^{-\frac{1}{2}}J_{2\mu}(u)J_{2\nu}(u)du$$

in the form of series of powers of α (>0). In the special case when $\mu+\nu$ is an integer, these series are not power series in the strict sense because their coefficients contain $\log\alpha$. *E. T. Copson* (St. Andrews).

Ikenberry, Ernest. A system of homogeneous spherical harmonics. Amer. Math. Monthly **62** (1955), 719–721.

Sato, Masachiyo. General formula of the Racah coefficient. Progr. Theoret. Phys. **13** (1955), 405–414.

The Racah coefficients were originally introduced in the theory of complex spectra; they arise in theoretical physics in problems involving coupling of angular momentum, and in mathematics in group-theoretical problems. $W(a, b, c, d; e, f)$ is of the form cw, where c^2 is a product of factorials and reciprocal factorials, and w is a terminating ${}_4F_3$ of unit argument. The author gives for ${}_4F_3$ a symbolical expression, and applies this to Racah coefficients with $e=3$ (Table 1), $e=7/2$ (Table 2), $e=4$ (Table 3), and $e=9/2$ (Table 4). *A. Erdélyi.*

See also: Fröberg, p. 412.

Ordinary Differential Equations

Minkevič, M. I. Geschlossene Integraltrichter bei verallgemeinerten dynamischen Systemen ohne Voraussetzung der Eindeutigkeit. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 73–88. (Russian)

Consider many-valued dynamical systems in a compact

metric space R defined thus: to each $p \in R$, $t \in (-\infty, +\infty)$ corresponds a closed set $S(p, t) \subset R$, $S(p, 0)=p$; to each set $F \subset R$ corresponds $S(F, t)=\mathbf{U}_{p \in F} S(p, t)$; $T_p=\mathbf{U}_t S(p, t)$ is called the funnel of point p; positive and negative funnels T_p^+, T_p^- are also obviously defined; $S(p, t)$ is called a section of T_p; p is called an uniqueness point to the right if there is an interval to the right of $t=0$ such that for each t in this interval $S(p, t)$ consists of just one point; if t_1 and t_2 have the same sign, $S[S(p, t_1), t_2]=S(p, t_1+t_2)$; if $q \in S(p, t)$, then $p \in S(q, -t)$; $S(p, t)$ is continuous in t in the sense of Hausdorff metric. In particular, differential systems $\dot{x}=f(x)$, x being an n-vector, f continuous, generate such a type of dynamical system. p_0 is a point of continuity if $S(p, t)$ is continuous at p_0 as a function of p in the sense of Hausdorff metric; $S(p, \tau)$ is called positively stable in the sense of Poisson if given $\varepsilon>0$, $T>0$, there is a $t \geqq T$ such that $d[S(p, \tau), S(p, t+\tau)]<\varepsilon$ (in particular, τ may be 0 and then we speak of stability of the point p); T_p is positively recurrent if given $\varepsilon>0$ there is an $l(\varepsilon)>0$ such that for any t', $t'' \geqq 0$ there is a τ, $0 \leqq \tau \leqq l(\varepsilon)$, such that $d[S(p, t'), S(p, t''+\tau)]<\varepsilon$. Theorems: if $S(p, t_0)$, $t_0 \geqq 0$, consists of points of continuity and is positively stable, each $S(p, \tau)$, $\tau \geqq t_0$, is positively stable; if T_p is positively recurrent, each $S(p, t)$, $t \geqq 0$, is positively stable. For each $q \in T_p^+$ define $t_q=\inf\{t: q \in S(p, t),\ t \geqq 0\}$; T_p is called positively closed if the set $\{t_q: q \in T_p^+\}$ is bounded and if for each $q \in T_p^+$ there is a (smallest) $t_q' \geqq 0$ such that $p \in S(q, t_q')$; it may be shown that the set $\{t_q': q \in T_p^+\}$ is also bounded. Theorems: for each point q of a closed T_p belonging to a dynamical system generated by a differential system there is a trajectory from q to p which does not coincide with any of the trajectories from p to q; if T_p is positively closed, for each $t' \geqq 0$ there is a $t'' \geqq t'$ such that $S(p, t') \subset S(p, t'+t'')$, if T_p is positively closed and $q \in T_p^+$, T_q is positively closed; if T_p is positively and negatively closed and $q \in T_p$ then $T_q=T_p$; if T_p is positively closed, each $q \in T_p^+$ is an ω-limit point and each ω-limit point belongs to T_p; assuming that the system is such that for each p no interval $(0, \tau)$ exists such that $p \in S(p, t)$ for all $t \in (0, \tau)$, then, if T_p is positively closed it is weakly positively periodic, i.e. there are $\tau \geqq 0$, $\sigma>0$, such that $t \geqq \tau$, $S(p, t+\sigma)=S(p, t)$; if the dynamical system is generated by a differential system of order 2, positive stability implies that T_p is positively closed and weakly periodic; if T_p is positively stable and strongly periodic ($\tau=0$ in the definition of weak periodicity) then $S(p, t)$, $t \geqq 0$, consists of just one point (there is positive uniqueness). *J. L. Massera* (Zbl 49, 209).

Baggis, G. F. Dynamical systems with stable structures. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 101–126. (Russian)

Translation of a paper by H. F. De Baggis in Contributions to the theory of nonlinear oscillations, vol. II, pp. 37–59 [Princeton Univ. Press, 1952; MR **14**, 557].

Kimura, Tosihusa. Sur une généralisation d'un théorème de Malmquist. III. Comment. Math. Univ. St. Paul. **4** (1955), 25–41.

A detailed study is given of the equation (1) $dy/dx=R(x, y)$, where $R(x, y)$ is rational in y, with coefficients analytic at the fixed singular point (s.p.) located at the origin. This is a continuation of the author's earlier work [same Comment. **3** (1955), 97–107; MR **16**, 820]. A certain special type (I) of (1) is introduced, admitting a logarithmic critical point at the origin. If (1) is not a Riccati equation, the problem is to find necessary and sufficient conditions so that (1) has integrals for which the origin is an essentially s.p., in whose vicinity there are no movable critical points. By a finite number of algebraic operations (1) is reduced to type (I). If (1) has an integral of a finite number of determinations for which the origin is an essentially s.p. without movable critical points in a vicinity of it, (1) must be a Riccati equation.

W. J. Trjitzinsky (Urbana, Ill.).

Skačkov, B. N. Qualitative picture of the behavior of the integral curves in the neighborhood of a singular point in one case. Vestnik Leningrad. Univ. **9** (1954), no. 8, 65–69. (Russian)

Take a real three-dimensional system

$$\dot{x}=Px+X(x) \tag{1}$$

where P is a constant matrix and the components of X are power series beginning with terms of degree at least two. Lyapunov has discussed the general case where P has one characteristic root zero and has singled out the special case where a regular transformation reduces (1) to the form

$$\dot{x}=\alpha x+\beta y+X,\ \dot{y}=\gamma x+\delta y+Y,\ \dot{z}=Z(x, y, z) \tag{2}$$

where the real parts of the characteristic roots of

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

have the same signs, and the vector (X, Y, Z) behaves like X in (1) save that the three components vanish for $x=y=0$. The z-axis is thus a line of singularities, there is a general solution $z=c+f(x, y, c)$ and the author discusses the character of the integral curves as the constant c varies. *S. Lefschetz* (Princeton, N.J.).

Petrovskiĭ, I. G., and Landis, E. M. On the number of limit cycles of the equation $dy/dx=P(x, y)/Q(x, y)$, where P and Q are polynomials of 2nd degree. Mat. Sb. N.S. **37(79)** (1955), 209–250. (Russian)

This is the full exposition of results announced earlier [Dokl. Akad. Nauk SSSR **102** (1955), 29–32; MR **16**, 1110]. *S. Lefschetz* (Princeton, N.J.).

Hukuhara, Masuo. Sur les polygones caractéristiques et le procédé de réduction au point singulier fixe ξ d'une équation différentielle ordinaire du premier ordre. Proc. Japan Acad. **31** (1955), 195–198.

Let $P(x, y)$, $Q(x, y)$ be polynomials in y of degree r, $r-2$, respectively, with coefficients which are regular functions of a positive rational power of x at $x=0$. Let $\mu \geqq 0$ be a rational number and $y=\gamma$ a zero of $P(0, y)$ or $Q(0, y)$. The author considers normal forms near $(x, y)=(0, \gamma)$ for the differential equation $x^{1+\mu}dy/dx=P(x, y)/Q(x, y)$. Detailed proofs are to appear elsewhere.

P. Hartman (Baltimore, Md.).

Andreev, A. F. Investigation of the behavior of the integral curves of a system of two differential equations in the neighborhood of a singular point. Vestnik Leningrad. Univ. **10** (1955), no. 8, 43–65. (Russian)

This is a detailed study of the local phase-portrait at the origin of a real system

$$\frac{dx}{dt}=y+X(x, y),\quad \frac{dy}{dt}=Y(x, y), \tag{1}$$

where X, Y are real convergent power series beginning with terms of degree at least two. Every analytical

system with the origin as isolated singularity, terms of the first degree not all zero but with both characteristic roots zero, may be reduced to the type (1) by a linear transformation of coordinates. Such a system was already investigated, strictly for stability, by Lyapunov [Mat. Sb. 17 (1893), 253–333=supplement (pp. 369–449) to "General problem of the stability of motion, Gostehizdat, Moscow, 1950; MR 12, 612]. The author applies extensively the method of Frommer [Math. Ann. 99 (1928), 222–272] and obtains essentially eight phase-portraits. He also gives analytical expressions for the trajectories tending to the origin. *S. Lefschetz* (Princeton, N.J.).

Pogorzelski, W. Problème aux limites de Poincaré généralisé. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 195–198.

Let D be a domain in the (x, y)-plane, L its frontier, (s) denoting points on L. A solution u is found of

$$\frac{du}{dn}+a(s)u=\lambda F\Big(s, u, \frac{du}{ds}\Big), \tag{1}$$

in the form of a logarithmic potential harmonic in D; L is a closed Jordan curve with a continuous tangent, the angle made by the latter being of class (H) (Hölder); $F(s, u, v)$ is defined for s on L, $|u|\leqq R_1$, $|v|\leqq R_2$ and satisfies conditions of type (H); $a(s)\in$ (H). The special case when $a(s)=0$, $\lambda F=f(s)-b(s)du/ds$, under conditions of analyticity, had been solved by H. Poincaré [Leçons de mécanique céleste, t. 3, Gauthier-Villars, Paris, 1910]; Poincaré's problem, with $a(s)$ possibly present and the data of class (H), had been treated by Hvedelidze [C.R. (Dokl.) Acad. Sci. URSS (N.S.) 30 (1941), 195–198; MR 2, 292]. The indicated generalization is stated very briefly; a full development is to be published later.

W. J. Trjitzinsky (Urbana, Ill.).

Feščenko, S. F. On the asymptotic decomposition of a system of linear differential equations. I. Ukrain. Mat. Ž. 7 (1955), 167–179. (Russian)

A study is made of the vector-matrix equation (1) $dx/dt=A(\tau, \xi)x$; the matrix $A(\tau, \xi)$ (of n^2 elements) has a suitable number of derivatives with respect to τ $(0\leqq\tau\leqq L)$; $A(\tau, \xi)=\sum_0^\infty \xi^s A^s(\tau)$, where $\tau=\xi^k t$, k is an integer $(1\leqq k<n)$, ξ is real; the roots of $\det|\lambda E-A^0(\tau)|=0$ (E being the identity matrix) are denoted by $\lambda_j(\tau)$ $(j=1, \cdots, n)$. The case considered is the one when for certain τ some of the $\lambda_j(\tau)$ are multiple. Specifically it is assumed that $\lambda_i(\tau)\neq\lambda_j(\tau)$ for $0\leqq\tau\leqq L$ and for $i\leqq r$, $j>r$. The solution of (1) is of form $x_1=U_1(\tau, \xi)\zeta_1+U_2(\tau, \xi)\zeta_2$, where $d\zeta_\nu/dt=A_\nu(\tau, \xi)\zeta_\nu$ $(\nu=1, 2)$ and the U_ν, A_ν are certain matrices having formal matrix expansions in powers of ξ; ζ_1, ζ_2 are vectors of r and $n-r$ elements, respectively. The author states that by means of these formal series the asymptotic decomposition of (1) can be realized into two systems of orders r and $n-r$.

W. J. Trjitzinsky (Urbana, Ill.).

Langer, Rudolph E. On the asymptotic forms of the solutions of ordinary linear differential equations of the third order in a region containing a turning point. Trans. Amer. Math. Soc. 80 (1955), 93–123.

The subject of this paper is the asymptotic study of a class of differential equations of the type

$$w'''+\lambda h_1(z, \lambda)w''+\lambda^2 h_2(z, \lambda)w'+\lambda^3 h_3(z, \lambda)w=0 \tag{1}$$

for large values of the complex parameter λ. The coefficients $h_j(z, \lambda)$ are supposed to be regular analytic in both variables for large λ and small z. At $z=0$ the equation (1) is assumed to have a "turning point," i.e. the multiplicity of the roots $\chi_j(x)$, $j=1, 2, 3$, of the "characteristic equation" $\chi^3+h_1(z, \infty)\chi^2+h_2(z, \infty)\chi+h_3(z, \infty)=0$ changes there. Specifically, $\chi_3(z)$ is to be regular at $x=0$, while $\chi_1(z)$ and $\chi_2(z)$ have a branch point there at which their values coincide. The author's aim is to find asymptotic approximations of any prescribed order $O(\lambda^{-r})$ to the solutions of (1) that are uniformly valid in domains containing the turning point. As a first step towards this goal (1) is changed into $L_1'(u)+p(z, \lambda)\lambda^{3-r}u=0$, where $L_1(u)$ is a linear second-order differential expression and $w=u\exp(\lambda\int_0^z \tau(z)dz)$ with a suitable $\tau(z)$. Since this new form differs formally little from $L_1'(u)=0$ the author can bring to bear on the problem some of the techniques he developed for second-order problems in Trans. Amer. Math. Soc. 67 (1949), 461–490 [MR 11, 438]. Starting from the equation for Bessel functions of order 1/3 he then constructs a differential equation of the form $L(y)=L_1(y)-q(z, \lambda)\lambda^{1-r}y=0$ with solutions that can be explicitly expressed in terms of such Bessel functions, and hence finds a "related equation"

$$L'(y)-g(z, \lambda)\lambda^{-r}L(y)=0 \tag{2}$$

with known fundamental systems of solutions. A comparison of the solutions of (1) and (2) by means of Volterra integral equations obtained by the method of variation of parameters shows that in certain specified sectors of the λ and z planes there exist fundamental systems of the two differential equations differing by terms of order $O(\lambda^{-r-1})$ except for common exponential factors. Every pair of values λ, z with $|\lambda|$ sufficiently large and $|z|$ sufficiently small lies in such a sector. Since all the z-sectors meet at $z=0$, the analytic continuations of these solutions around $z=0$ can be asymptotically evaluated by determining at $z=0$ the linear relations that must exist between any two such fundamental systems. *W. Wasow.*

Langer, Rudolph E. The solutions of the differential equation

$$v'''+\lambda^2 zv'+3\mu\lambda^2 v=0.$$

Duke Math. J. 22 (1955), 525–541.

The equation $v'''+\lambda^2 zv'+3\mu\lambda^2 v=0$, where μ is arbitrary and λ a complex parameter of large absolute value, has a double turning point at $z=0$. The author derives asymptotic formulae for the solutions at this turning point. First, the principal solutions (1) $v_j(z)$ $(j=0, 1, 2)$ are obtained as power series in z. Then for each integral index q a set (2) $v_{q,j}(z)$ of formal solutions in powers of $1/\lambda$ is established which represents actual solutions asymptotically in the sector $(q-1)\pi\leqq\arg\xi\leqq q\pi$ $(\xi=\frac{1}{3}i\lambda z^{3/2})$ of the z-plane. Between the solutions of the set (1) and those of any set (2) there exist linear relations with coefficients that are independent of z. The main results of this paper consist in the determination of these relations.

M. Zlámal (Brno).

Jeffreys, Harold. On the use of asymptotic approximations of Green's type when the coefficient has zeros. Proc. Cambridge Philos. Soc. 52 (1956), 61–66.

The author continues the study of differential equations of the form (*) $y''=h^2\chi(x)y$ in which h is a large parameter and $\chi(x)$ may have zeros. On either side of a zero, asymptotic solutions of (*) may be obtained by the Liouville-Green method, and the connection across the zero of $\chi(x)$ may be established by a method due to the author

[H. Jeffreys and B. S. Jeffreys, Methods of mathematical physics, 2nd ed., Cambridge, 1950; MR 12, 12]. This leads to uniform appraisals of the error involved in the asymptotic approximation in closed intervals excluding zeros of $\chi(x)$. In the present paper the author applies his asymptotic theory to the solution of boundary-value problems, it being assumed that $\chi(x)$ does not vanish at the end points of the interval. The following cases are discussed: (i) $-1\leqq x\leqq 1$, $\chi(x)$ has the same sign as x; (ii) $-\infty<x<\infty$, $\chi(x)$ positive for $|x|<1$, and negative for $|x|>1$; (iii) $-\infty<x<\infty$, $\chi(x)$ negative for $|x|<1$ and positive for $|x|>1$; (iv) the one-dimensional Schrödinger equation.

In addition to obtaining asymptotic results in the above cases the author gives some discussion clarifying the relation between his approach and that developed by Langer. *A. Erdélyi* (Pasadena, Calif.).

Lewis, D. C. On the perturbation of a periodic solution when the variational system has non-trivial periodic solutions. J. Rational Mech. Anal. **4** (1955), 795–815.

In a previous paper [Duke Math. J. **22** (1955), 39–56; MR **17**, 38] the author determined bounds for the interval of values of the parameter μ for which periodic solutions exist, under the assumption that the variational equations do not have non-trivial periodic solutions. The same problem is here considered when this assumption is not satisfied. Let

$$\dot{x}=A(t,\mu)x+\mu h(t,\mu)+\sigma q(x,\mu,t)+p(x,\mu,t); \tag{5}$$

A is a continuous $n\times n$ matrix, analytic in μ for $|\mu|\leqq\lambda$; h, q, p are vectors of class C' for $|\mu|\leqq\lambda$, $\|x\|\leqq r$; as functions of t they are all periodic of period 2π; $q(y\mu,\mu,t)=\mu^2Q(y,\mu,t)$, $p(y\mu,\mu,t)=\mu^3P(y,\mu,t)$, P, Q being of class C'; if $\Xi^*(s)$ is the $k\times n$ matrix whose rows form a complete set of periodic independent solutions of the adjoint system $d\xi/ds=-\xi A(s,0)$, assume that $\int_0^{2\pi}\Xi^*(s)h(s,0)ds=0$; assume moreover that any nonsingular matrix $X(t,\mu)$, analytic in μ, solution of $\dot{X}=AX$, satisfies $\det[X(0,\mu)-X(2\pi,\mu)]\neq 0$ for $0<|\mu|\leqq\lambda$ and furthermore

$$[X(0,\mu)-X(2\pi,\mu)]^{-1}=\mu^{-1}M+N(\mu),$$

M constant, N analytic. Under these assumptions, it is possible to find explicitly numbers $L>0$, $\gamma>0$, such that for $|\mu|\leqq L$, $|\sigma|\leqq\gamma$ there is a unique continuous periodic solution $x(t,\mu,\sigma)$ of (5) such that $\|x\|\leqq|\mu|r\lambda^{-1}$. Refinements in the assumptions, other uniqueness questions, etc. are also considered. *J. L. Massera* (Montevideo).

Hirasawa, Yoshikazu. On singular perturbation problems of non-linear systems of differential equations. Comment. Math. Univ. St. Paul. **3** (1955), 115–122.

The convergence as $\varepsilon\to 0$ of solutions of the system $\dot{x}=f(x,y,t,\varepsilon)$, $\varepsilon\dot{y}=g(x,y,t,\varepsilon)$ to those of the "degenerate" system $\dot{x}=f(x,y,t,0)$, $0=g(x,y,t,0)$ is studied. For previous results, compare J. J. Levin and N. Levinson [J. Rational Mech. Anal. **3** (1954), 247–270; MR **15**, 795]. The smoothness hypotheses on f and g are weaker than is customary (Lipschitz conditions in place of continuous differentiability); hypotheses relating to the characteristic roots of the matrix $\|\partial g_i/\partial y_j\|$, however, are replaced by an assumption that the jth component of $g(x,y,t,0)$ depends only on x, t and the jth component of y, and that

$$[g(x,y^*,t,0)-g(x,y,t,0)]/(y_j^*-y_j)\leqq -C<0.$$

G. E. H. Reuter (Manchester).

Hirasawa, Yoshikazu. On singular perturbation problems of non-linear systems of differential equations. II. Comment. Math. Univ. St. Paul. **4** (1955), 15–23.

The same problem as in part I is treated by a different method, under considerably weaker assumptions (unfortunately too lengthy to state here). *G. E. H. Reuter.*

Požarickiĭ, G. K. On the characteristic number of the vanishing solution of the equations of perturbed motion. Prikl. Mat. Meh. **19** (1955), 481–484. (Russian)

The author makes various simple remarks on the connection between the characteristic numbers of the linear vector system $dx/dt=P(t)x$ and those of the perturbed system $dx/dt=P(t)x+X(x,t)$, where X is holomorphic in x and is $O(\|x\|^2)$ for small $\|x\|$. An application is to whether the vector system $dy/dt=Y(y,t)$ can have two solutions whose difference tends to zero as $t\to\infty$. Another application is to a dynamical problem, but the explanation is incomplete. [Reviewer's remark: The author mostly assumes that $dx/dt=P(t)x$ is "regular", for which case fuller information is to be had from series solutions; see, e.g., V. V. Nemyckiĭ and V. V. Stepanov, Qualitative theory of differential equations, 2nd ed. Gostehizdat, Moscow-Leningrad, 1949, pp. 240–241; for a review of the 1st ed. see MR **10**, 612.] *F. V. Atkinson.*

Taam, Choy-tak. Criteria of boundedness of the solutions of nonlinear differential equations. Proc. Amer. Math. Soc. **6**(1955), 377–385.

Sufficient conditions for the boundedness of solutions of

$$\frac{d}{dt}\left(m(t)\frac{dx}{dt}\right)+\sum_{i=1}^{k}f_i(t)x^{2i-1}=0,$$

distinct from those already found by the author [same Proc. **5** (1954), 705–715; MR **16**, 701], are given. For the more general equation

$$\frac{d}{dt}\left(m(t)\frac{dx}{dt}\right)+\sum_{i=1}^{n}g_i(t)x^i=0$$

(n being an odd integer) $x(t)$ and $m(t)dx/dt$ are bounded on (a,∞) provided that $d(m(t)g_i(t))/dt$ is in $L(a,\infty)$ for $i=1,\cdots,n$ and $m(t)g_n(t)$ has a positive lower bound on (a,∞). *G. E. H. Reuter* (Manchester).

Erugin, N. P. Qualitative methods in the theory of stability. Prikl. Mat. Meh. **19** (1955), 599–616. (Russian)

This is an expository paper on the work done by Soviet scientists, mostly along the line of Lyapunov's second method, between 1950 and 1954. There are 35 references. *S. Lefschetz* (Princeton, N.J.).

Bieniek, Maciej. Methods of the theory of stability of motion. Rozprawy Inż. **3** (1955), 325–358. (Polish. Russian and English summaries)

The paper gives a clear, simple presentation of Liapounoff's theory of stability of motion. It contains no new results, but discusses the following well-known problems: Liapounoff's definition of stability; basic concepts; general formulation of problem; Liapounoff's theorems for steady and unsteady motion; the stability theorems in the first approximations; the criteria due to Routh and Hurwitz and to Nyquist; examples are drawn from the field of mechanical and electrical engineering (regulators for engines, triode resonator). References mostly drawn from Russian literature. *J. Kestin.*

Razumihin, B. S. On stability of the trivial solution of systems of second order. Prikl. Mat. Meh. **19** (1955), 279–286. (Russian)

The system is $\dot{x}_1=p_{11}(x_1, x_2)x_1+p_{12}(x_1, x_2)x_2$, $\dot{x}_2=x_1$, where p_{11} and p_{12} are continuous. The author gives a geometrical condition on p_{11} and p_{12} to ensure that a positive-definite quadratic form $V(x_1, x_2)$ should be a Lyapunov function, such that $dV/dt\leqq 0$; this involves considering parabolas in the (p_{11}, p_{12})-plane. The known result is derived that "the trivial solution is stable in the large" if const $<p_{11}<$ const <0, $p_{12}=$ const <0. A development of the method is applied to cases where $p_{11}<0$, $p_{12}<0$ and p_{11} and p_{12} depend either on x_1 only or x_2 only. The author is correct in asserting Lyapunov stability in these cases; whether he means "stability in the large" is not clear. Finally he considers the system $\dot{x}_1=p_{11}(x_1, x_2)x_1+p_{12}(x_1, x_2)x_2$, $\dot{x}_2=x_1\pm x_2$ in a similar way.

F. V. Atkinson (Canberra).

Pliss, V. A. A connection between the region of stability and the region of uniform analyticity in the initial data of the solutions of a system of differential equations. Vestnik Leningrad. Univ. **9** (1954), no. 11, 19–32. (Russian)

Let $z=(z_1, \cdots, z_n)$ denote a complex n-vector and let R be its space. Set $(m)=(m_1, \cdots, m_n)$, $m_i>0$, $z^{(m)}=z_1^{m_1}\cdots z_n^{m_n}$ and consider the equation

$$\dot{z}=Z(z, t)=\sum P^{(m)}(t)z^{(m)}, \tag{1}$$

where the P's are continuous functions of the real variable t for $t\geqq 0$ and for all such t the sum converges throughout R. If c is an initial point, then for a region $\|c\|<a$ and $0\leqq t<T$ there is a solution due to Lyapunov

$$f(c, t)=\sum L^{(m)}(t)c^{(m)}, \tag{2}$$

where the L's are differentiable functions of the time. Let $E_T\subset R$ be the set of points reached by that solution for some $t\in[0, T)$. Theorem 1. E_T is the neighborhood of the origin consisting of the points c for which $f(c, t)$ is analytic for every $c\in[0, T)$.

The various types of stability (in the complex sense) may be defined for (1). Assume now that (1) is uniformly asymptotically stable (u.a.s.) (i.e. with constants independent of t) in $[0, +\infty]$. If so, there will exist a region $\Pi(a):|c_k|<a_k$ $(k=1, 2, \cdots, n)$ such $f(c, t)$ is bounded in $\Pi(a)$ for all t and that $f(c; t)\to 0$ as $t\to+\infty$. Theorem 2. A necessary and sufficient condition that the situation just described hold is that every $L^{(m)}(t)\to 0$ as $t\to+\infty$.

Let now $S'\subset R$ be the set of points c such that $f(c, t)\to 0$ as $t\to+\infty$. Let S be the interior of S'. Every neighborhood of a boundary point of S contains a point c such that $f(c, t)$ does not $\to 0$ as $t\to\infty$. Theorem 3. If $c\in S'$, then $f(c, t)$ is asymptotically stable (a.s.) relative to the neighboring solutions of (1).

A function $F(z, t)$ is said to be uniformly analytic in a region U of R if its Taylor series about any $z^0\in U$ is uniformly convergent for $\|z-z^0\|<\varepsilon$ and $t\in[0, +\infty]$. Theorem 4. If (1) is u.a.s. at the origin, then the set S is the uniformly analytic domain of $f(c, t)$ which contains the origin. Corollary 1. The series (2) converge uniformly in the largest $\Pi(a)$ contained in the stability region. At any point outside $\Pi(a)$ at least one of the component series is not uniformly convergent relative to $t\geqq 0$. Corollary 2. Let the origin be u.a.s. Then a necessary and sufficient condition that it be a.s. in the large (i.e. that $S'=R$) is that the series (2) converge uniformly in any bounded domain of R and for all $t\geqq 0$. Corollary 3. At any boundary point of S, $f(c, t)$ is unbounded.

Consider now a system $\dot{z}=Az+Z(z)$ where A is a constant matrix and Z has the form (2) with P's constant and no linear terms. Let S be the domain of asymptotic stability (i.e. from which all trajectories lead to the origin). Theorem 5. If the characteristic roots of A all have negative real parts then S is not bounded.

S. Lefschetz (Princeton, N.J.).

Krasovskiĭ, N. N. On stability in the first approximation. Prikl. Mat. Meh. **19** (1955), 516–530. (Russian)

This paper contains a searching study of a real n-vector system

$$\dot{x}=X(x, t), \text{ in } \Omega\colon |x_i|\leqq H,\ 0\leqq t, \tag{1}$$

and the relations between stability and Lyapunov functions v. It is assumed that in Ω: $\partial X_i/\partial x_j<L$, $X(0, t)=0$. The following conditions play an important role: If $p\in\Omega$, then the arc of solution through p at time t_0 in Ω beyond t_0 is denoted by γ_p^+, and at time t before t_0 by γ_p^-; $r(p)=$the Euclidean distance of p from the origin; $r(p, t_0, t)$ the same for the point on γ_p^+ or γ_p^-. A first condition is: There exist B, $\alpha>0$ such that on γ_p^+ or γ_p^-:

$$\log r(p, t_0, t)\geqq \log Br(p)+\alpha|t-t_0|. \tag{2}$$

A second set of conditions is:

$$|v|\leqq c_1 r^A,\ \frac{dv}{dt}\geqq c_2 r^A,\ \left|\frac{\partial v}{\partial x_i}\right|<c_3 r^{A-1}, \tag{3}$$

where c_1, c_2, $c_3 A$ are positive constants and dv/dt is the total time derivative along a trajectory.

Theorem 1: If (2) holds in a certain Ω then there exists in Ω a $v(x, t)$ of class C^1 satisfying (3), and conversely. Theorem 2: If the origin is asymptotically stable and (2) holds then there exists a v satisfying the Lyapunov conditions on asymptotic stability and also (3). For instability one only needs to consider (2) for γ_p^+. Let it be denoted by (2′). Theorem 3: If (2′) holds, then there is a Lyapunov function v satisfying his second instability theorem and such that (3) holds.

With (1) consider

$$\dot{x}=X(x, t)+R(x, t) \tag{4}$$

and suppose that

$$|R_i|<a(\sum x_i^2)^{\frac{1}{2}}. \tag{5}$$

Theorem 4. For a suitably small a the system (4) will have the same asymptotic stability or instability behavior as (1). Corollary 1. Let (1) be linear and let

$$|R_i|<M(\sum x_i^2)^{\frac{1}{2}+\beta};\ M,\ \beta>0.$$

Then R does not affect the stability properties stated in Theorem 4. Similarly if (2) holds there is an $h>0$ such that the stability properties in question for (1) are also those of the equations in finite differences

$$x^{(k+1)}-x^{(k)}=X(x^{(k)}, kh)^k$$

or of the mixed system

$$\dot{x}_i=X_i(x_1(t-\eta_{i1}), \cdots, x_n(t-\eta_{in}), t)$$

for all $|\eta_{ij}|<h$.

Consider the conditions analogous to (3):

(6) $|v|<c_1r^A$, $\frac{dv}{dt}>c_2r^{A+m-1}$, $\left|\frac{\partial v}{\partial x_i}\right|<c_3r^{A-1}$,

(7) $|R_i(x, t)|<a(\sum|x_i|)^m$

in relation to the systems

(8) $\dot{x}=P(x)$,

(9) $\dot{x}=P(x)+R(x, t)$, $R(0, t)=0$,

where the $P_i(x)$ are polynomials without constant terms and of degree $\leq m$. Suppose also that (8) has no solution bounded for all t. Then: Theorem 5: There exists a $v(x, t)$ of class C^1 such that (6) holds. Furthermore, if (7) holds with suitably small a, (9) behaves like (8) as regards the stability properties of Theorem 4.

Further stability comparison theorems, which we leave out, are given in the paper. *S. Lefschetz.*

Staržinskiĭ, V. M. On stability of unsteady motions in a special case. Prikl. Mat. Meh. **19** (1955), 471–480. (Russian)

The solutions of $\dddot{x}+p\ddot{x}+\dot{x}+r(t)x=0$, with $p=$ const >0, $0\leq r(t)\leq N$, are bounded as $t\to\infty$ if $N\leq p-\frac{1}{4}p^3, 0<p\leq\sqrt{2}$, or if $N\leq p^{-1}$, $p\geq\sqrt{2}$. This supplements a previous result of the author [Prikl. Mat. Meh. **16** (1952), 500–504; MR **14**, 170]. In the case of $\dddot{x}+p\ddot{x}+q\dot{x}+s(t)x=0$, with constant positive p and q, and $0\leq s(t)\leq\Sigma$, sufficient conditions are $(\frac{1}{4}p^3-pq+2)p\leq\Sigma\leq 1/q$ if

$$(8p)^{-1}(p^3+8+(p^6+16p^3)^{\frac{1}{2}})\leq q\leq\tfrac{1}{4}p^2+2p^{-1},$$

and $\Sigma\leq 1/q$, if $q\geq\frac{1}{4}p^2+2p^{-1}$. The method of proof is to establish the existence of a positive-definite quadratic form which serves as a Lyapunov function.

F. V. Atkinson (Canberra).

★ **Bogolyubov, N. N., i Mitropol'skiĭ, Yu. A. Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ. [Asymptotic methods in the theory of nonlinear oscillations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 449 pp. 13.40 rubles.

The present book is the fourth or fifth major treatise published in recent years by Soviet scientists on the general topic of non-linear oscillations, which serves to indicate the great value which is attached in the USSR to this general topic. The general program of the book is not too far from the program of the 1937 Krylov-Bogolyubov monograph [Introduction to non-linear mechanics, Izdat. Akad. Nauk SSSR, Kiev, 1937; see MR **4**, 142]. However, although the book is addressed primarily to physicists and engineers, its mathematical treatment is most careful, which was by no means the case with the 1937 monograph. The book is also much more orderly and most readable: an excellent contribution in every respect.

Let us pass now to a more detailed description. The book consists of a lengthy introduction and five chapters.

Introduction. One finds here a large number of examples: physical, mechanical, electrical, illustrating the appearance of non-linear oscillations in various domains. The description is always careful and clear.

Chapter I: Self-oscillations in nearly linear systems. The prototype is

(1) $\ddot{x}+\omega^2x=\varepsilon f(x, \dot{x})$, $\varepsilon\geq 0$ and small.

The basic solution

(2) $x=a\cos\psi=a\cos(\omega t+\varphi)$

is used as a starter and one tries to find a solution for (1), with a, ψ variable, of the form

(3) $x=a\cos\psi+\varepsilon u_1(a, \psi)+\varepsilon^2u_2(a, \psi)+\cdots$,

where the u_n are periodic with period 2π in ψ and

(4) $\dot{a}=\varepsilon A_1(a)+\varepsilon^2A_2(a)+\cdots$,

(5) $\dot{\psi}=\omega+\varepsilon B_1(a)+\varepsilon^2B_2(a)+\cdots$

(slowly varying amplitude and phase). Upon dropping everywhere the terms of degree $>m$, one obtains the mth approximation. The solutions are found by substituting in (1) and identifying like powers of ε yielding an infinite system of differential equations. The arbitrariness involved is settled by imposing

(6) $\int_0^{2\pi}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}\cos\psi\,d\psi=\int_0^{2\pi}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}\sin\psi\,d\psi=0.$

A number of special cases are treated at length.

Chapter II: Method of the phase plane. This is familiar ground. Of particular interest is the treatment of relaxation oscillations, including the Dorodnytsin treatment of the approximation for the period and amplitude for large values of the parameter (like the one in van der Pol's equation).

Chapter III: Influence of forced oscillations. Here the f of (1) is of the form $f(\nu, t, x, \dot{x})$, with ν as the imposed frequency, and f with period $2\pi i\nu t$. More precisely,

$$f=\sum_{n=-N}^{+N}e^{in\nu t}f_n(x, \dot{x}).$$

Careful distinction is made between resonance and non-resonance. The attempted solution is again (3), under conditions (4), (5), and with u_n now of the form $u_n(a, \psi, \nu t)$.

Chapter IV: Averaging method. The system studied is

(7) $\dot{x}=\varepsilon X(t, x)$

where x, X are n-vectors and the treatment is by vector-matrix throughout. Let

$$X_0(\zeta)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T X(t, \zeta)dt,$$

and let ζ be a solution of $\dot{x}=\varepsilon X_0(\zeta)$. Then ζ is considered as the first approximation and from it one passes to higher approximations by using suitable Fourier series expansions.

Chapter V: Foundation of the asymptotic methods. In substance it is shown rigorously that under certain very general assumptions, the approximation methods of the previous chapters are justified.

The volume terminates with an all too short bibliography and there is no index. *S. Lefschetz.*

Kuzovkov, N. T. Investigation of the stability of systems having undamped oscillating links by the method of logarithmic frequency characteristics. Moskov. Gos. Univ. Uč. Zap. **172** (1954), Meh. **5**, 207–213. (Russian)

Bellman, Richard. Perturbation methods applied to nonlinear dynamics. J. Appl. Mech. **22** (1955), 500–502.

The Lindstedt and the Lighthill techniques in the theory of non-linear differential equations are discussed. A modification of the techniques is illustrated by examples with a reference to the author's previous papers. In particular, the van der Pol equation, $x''+\lambda(x^2-1)x'+x=0$, $\lambda>0$, is treated by the Lindstedt techniques, although, as the author remarks, there is as yet no theoretical justification

for the success of the method in this type of equation. The results obtained are comparable to those obtained by van der Pol using a different method. *S. Kulik.*

Aymerich, Guiseppe. Oscillazioni periodiche di un sistema di Rocard a due gradi di libertà nel caso di quasi risonanza. Rend. Sem. Fac. Sci. Univ. Cagliari **24** (1954), 177–186 (1955).

The same system previously studied by the author [same Rend. **24** (1954), 51–62; MR **17**, 39] is considered now in the case where $\omega_1-\omega_2=\varepsilon\sigma(\omega_1+\omega_2)/2$. Two or six harmonic vibrations are possible; sufficient conditions for their stability are briefly discussed. As in the previous paper standard methods of non-linear mechanics are used. *J. L. Massera* (Montevideo).

Lundquist, Stig. Subharmonic oscillations in a nonlinear system with positive damping. Quart. Appl. Math. **13**, 305–310 (1955).

The equation

$$(*) \qquad y''+y=\varepsilon\{F\sin(px+\alpha)-ky'-f(y)\}$$

is studied with ε small. Existence and stability of periodic solutions of (*) including subharmonics is studied using the first approximation only with a technique due to Cartwright [Contributions to the theory of nonlinear oscillations, Princeton, 1950, pp. 149–241; MR **11**, 722]. Comparison is made with experimental results. *W. S. Loud* (Cambridge, Mass.).

★**Hukuo, Nobuhei. Transient oscillations of nonlinear systems.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 611–614. Science Council of Japan, Tokyo, 1952.

The author discusses two numerical examples of Duffing's equation with square-wave forcing term of period T by the use of what he terms the transient diagram, which he obtains by connecting the points $(x(t_0+nT), \dot{x}(t_0+nT))$ in the phase plane. *H. A. Antosiewicz* (Washington, D.C.).

★**Meerov, M. V. Investigation and construction of multiple-contour systems with many regulating quantities having a high degree of consistent accuracy.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 230–241. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

The mathematical investigation reduces to determining the asymptotic behavior of a system of linear equations with constant coefficients having the form

$$[D_1(p)M_1(p)+K_1]x_1=-K_1'M_1(p)[a_{12}x_2+\cdots+a_{1n}x_n],$$
$$[D_2(p)M_2(p)+K_2]x_2=-K_2'M_2(p)[a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n]$$
$$\cdots \qquad \cdots$$
$$[D_n(p)M_n(p)+K_n]x_n=-K_n'M_n(p)[a_{n1}x_1+\cdots+a_{n,n-1}x_{n-1}],$$

where $D_n(p)$ and $M_n(p)$ are polynomials in $p=d/dt$. *R. Bellman* (Santa Monica, Calif.).

Čermožskaya, L. N. On the influence of the tuning of the parameters of an automat on the self-oscillations of a system of regulation. Moskov. Gos. Univ. Uč. Zap. **172** (1954), Meh. **5**, 191–205. (Russian)

Let φ be an angular displacement variable for a controlled object and η the variable of the control acted upon by the servo. These variables satisfy a system of the third order of the form

$$(1) \qquad \begin{aligned} T^2\ddot{\varphi}&=-U\dot{\varphi}-k\varphi-\eta+\chi,\\ W\dot{\eta}&=h\psi+\mu F(\psi),\\ \psi&=Y^2\ddot{\varphi}+X\dot{\varphi}+l\varphi-m\eta. \end{aligned}$$

Here the new letters except χ, ψ, F are positive constants, with μ small; χ is the exterior disturbance and $h\psi+F(\psi)$ the servo characteristic. In particular, F contains no linear terms. Upon eliminating η the system (1) is replaced by one of order three of the form

$$(2) \qquad a(D)\varphi+b(D)\psi=\chi,\ c(D)\varphi+d(D)\psi=-\mu F,\ D=d/dt,$$

where $a, \cdots, d$ are polynomials with constant coefficients.

In a system such as the present stable oscillations are highly undesirable unless they are of very small amplitudes. The author studies first the linear case: $\mu=0$. Conditions are established for the existence of a pair of pure complex characteristic roots and so of an oscillation. By Poincaré's well known method there is obtained from this solution one for $\mu\neq0$, a process due to Bulgakov being followed in the calculations. Certain stability ranges for the periodic solution are also considered. *S. Lefschetz.*

Kuzovkov, N. T. The formation and transformation of structure schemes for systems of automatic regulation. Moskov. Gos. Univ. Uč. Zap. **172** (1954), Meh. **5**, 215–225. (Russian)

Blehman, I. I., and Džanelidze, G. Yu. Dynamics of the Bouasse-Sardou regulator. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 10, 48–59. (Russian)

The differential equations,

$$I\ddot{\theta}+k\dot{\theta}=Mgr+\chi e(x-e\sin\theta)\cos\theta,$$
$$m\ddot{x}+\beta\dot{x}+\chi x=mg+\chi e\sin\theta$$

of the dynamics of the Bouasse-Sardou regulator are analysed by the Poincaré-Liapounoff methods. The regulator was constructed to enable the lowering at a constant speed of a load attached to a rope. It was previously described by Y. Rocard, [Dynamique générale des vibrations, 2nd ed., Masson, Paris, 1949; for a review of the 1st ed. see MR **10**, 628]. The authors show that Rocard's results, obtained by a non-rigorous method, give, in general, a true picture of the motion, although some essential corrections are necessary. The analysis of the character of the movement is given by the authors in more detail. Because the speed of the movement of the load in the stationary position is not strictly constant, a formula has been derived for determining the degree of non-uniformity of the motion. *S. Kulik.*

Vodička, Václav. Steady temperature in multilayer bodies. Appl. Sci. Res. A. **5** (1955), 321–326.

The author obtains a convenient form of the solution of the system of differential equations $u_i''(x)+\omega x^{-1}u_i(x)=0$ $(x_i<x<x_{i+1};\ i=1, 2, \cdots, n)$, subject to boundary conditions $-K_1u_1'(x_1)+u_1(x_1)=A$, $K_nu_n'(x_{n+1})+u_n(x_{n+1})=B$, and to these conditions at $x=x_{i+1}$ $(i=1, 2, \cdots, n-1)$: $u_i'=(u_{i+1}-u_i)h_{i+1}$, $u_{i+1}'=(u_{i+1}-u_i)h_{i+1}'$. Except for u_i and x_i, all letters in the above problem denote constants. In the special case $\omega=0$, $u(x)$ represents steady-state temperatures in a composite slab. When $\omega=1$, $u(x)$ represents corresponding temperatures in a composite hollow cylinder of infinite length, where x is distance from the axis. When $\omega=2$, $u(x)$ represents temperatures at

distance x from the center in a composite hollow sphere. In all cases the above interface conditions at $x=x_{i+1}$ ($i=1, 2, \cdots, n-1$) represent linear surface heat-transfer conditions, rather than the usual interface conditions.

R. V. Churchill (Ann Arbor, Mich.).

★ **Belyustina, L. N. On an equation from the theory of electrical machines.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 173–186. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

A qualitative study is made of the solutions of the nonlinear differential equation

$$\frac{d^2\theta}{dt^2}+k\frac{d\theta}{dt}+\sin\theta+r\sin 2\theta=T.$$

This differential equation is related to the oscillation of the rotor of a synchronous motor with asynchronous starting. The angle between the field of the rotor and the revolving magnetic field of the stator is the term θ. The constant k depends on friction and electrical damping. The term T represents the external loading of the motor and is assumed to be a constant. The sine terms approximate the moment due to the angle between the rotor and stator fields.

The phase space natural to this problem is the surface of a cylinder. The existence and stability of singular points and limit cycles on the cylinder are studied. It is shown that the only stationary modes of operation of the motor are: (1) synchronous rotation without vibration and (2) rotation which varies in magnitude but with a constant direction of sliding. The first corresponds to a stable singular point. The second corresponds to a stable limit cycle encircling the cylinder.

It is shown that for some values of the parameters k, r, and T, some initial conditions give rise to (1) and other initial conditions lead to (2). Conditions on the parameters are also given under which all initial conditions lead to (1).

J. P. LaSalle (Notre Dame, Ind.).

Dikiĭ, L. A. The zeta function of an ordinary differential equation on a finite interval. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 187–200. (Russian)

Let λ_n be the eigenvalues and $\phi_n(x)$ the eigenfunctions of the problem: $u''+p(x)u=\lambda u$, $u(0)=u(\pi)=0$. The functions $Z(s)=\sum\lambda_n^{-s}$ and $Z(s;x,y)=\sum\lambda_n^{-s}\phi_n(x)\phi_n(y)$ are studied. Use is made of the asymptotic formulas for λ_n and ϕ_n.

N. Levinson (Cambridge, Mass.).

Neĭgauz, M. G. On determination of the asymptotic behavior of a function $q(x)$ by properties of the spectral function of the operator $-y''+q(x)y$. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 25–28. (Russian)

Let the spectral function associated with $y''+(\lambda-q(x))y=0$, $y(0)=0$, $y'(0)=1$ for $0\leq x<\infty$ be $\varrho(\lambda)$. The determination of $q(x)$ when $\varrho(\lambda)$ is given has been carried out by Gelfand and Levitan. Here sufficient conditions are given on $\varrho(\lambda)$ to imply that (1) $q(x)$ is algebraically small as $x\to\infty$, and (2) that $q(x)$ vanishes identically for large x.

N. Levinson (Cambridge, Mass.).

Putnam, C. R. Necessary and sufficient conditions for the existence of negative spectra. Quart. Appl. Math. **13** (1955), 335–337.

An extension [P. Hartman and A. Wintner, Amer. J. Math. **73** (1951), 885–980; MR **13**, 652] of a theorem of Liapounoff states that if $f(t)$ is continuous on $0\leq t\leq 2T$, then the inequality $\int_0^{2T} t(2T-t)f^+(t)dt\leq 2T$, where $f^+=\max(0, f)$, implies that a solution $x=x(t)\not\equiv 0$ of $x''+fx=0$ has at most one zero on $0\leq t\leq 2T$; the constant 2 cannot be replaced by $2+\varepsilon$ (>2) in this assertion. The author shows that this statement has a partial converse: if $\int_0^{2T} t^2(f(t)+f(2T-t))dt>2T$, then, for some $\lambda<0$, the equation $x''+(\lambda+f)x=0$ has a solution $x=x(t)\not\equiv 0$ with two zeros on $0\leq t\leq 2T$, the constant 2 cannot be replaced by $2-\varepsilon$ (<2). The proof depends on Parseval's relation belonging to the Sturm-Liouville problem $x(0)=x(2T)=0$.

P. Hartman (Baltimore, Md.).

Schwarz, Binyamin. Complex nonoscillation theorems and criteria of univalence. Trans. Amer. Math. Soc. **80** (1955), 159–186.

If $p(z)$ is regular in E: $|z|<1$, and $f(z)=u(z)/v(z)$ is the quotient of two linearly independent solutions $u(z)$ and $v(z)$ of the equation (*) $y''+p(z)y=0$, then $f(z)$ is meromorphic in E with simple poles and $f'(z)\neq 0$ (=meromorphic function of the restricted class). Here the Schwarzian derivative, $\{f(z), z\}=2p(z)$, is regular in E; conversely the Schwarzian derivative of a function of the restricted class is always regular in E. Z. Nehari [Bull. Amer. Math. Soc. **55** (1949), 545–551; MR **10**, 696] gave a necessary condition on $p(z)$ in order that $f(z)$ be univalent in E as well as a somewhat stronger sufficient condition, both sharp. As an extension of Nehari's sufficient condition the author proves that if $|p(z)|<[1-|z|^2]^{-2}$ in the annulus $0<x_0<|z|<1$, then the equation (*) is nonoscillatory in E, that is, every solution has only a finite number of zeros in E. Again this limit is sharp as the author concludes from an example given by the reviewer in a note [ibid. **55** (1949), 552–553; MR **10**, 697] following Nehari's paper. Similarly, if $f(z)$ is meromorphic in E and $|\{f(z), z\}|<2[1-|z|^2]^{-2}$ for $x_0<|z|<1$, then $f(z)$ is finitely-valent in E and here 2 is the best constant. By conformal mapping these results are now extended, first to simply-connected domains, then via the universal covering surface to multiply-connected ones. The conditions for nonoscillatory character in half-planes and strips are rather simple. These results contain as special cases some other results of Nehari [Amer. J. Math. **76** (1954), 689–697; MR **16**, 131]. The rest of the paper is devoted to a study of the relations that hold between bounds on $|p(z)|$ in E on one hand and the Euclidean or non-Euclidean minimum distance between the zeros of solutions of (*) in E on the other. Thus if $|p(z)|<a[1-|z|^2]^{-2}$, $a>1$, in E, then the non-Euclidean distance between any two zeros of any given solution of (*) exceeds $\log(a^{\frac{1}{2}}+1)(a^{\frac{1}{2}}-1)^{-1}$. Conversely, if this quantity is the lower bound of the distance between zeros of any solution in E, then $|p(z)|<3a[1-|z|^2]^{-2}$, and in both cases the estimates are asymptotically the best possible as $a\to\infty$. Finally the author obtains an upper bound for the valency of a meromorphic function of the restricted class in terms of the quantity x_0 introduced above.

E. Hille (New Haven, Conn.).

See also: Epstein, p. 343; Feldman, p. 361; Slugin, p. 387; Židkov, p. 394; Dennis and Poots, p. 412; Hil'mi, p. 447.

Partial Differential Equations

★ **Tricomi, F. G. Sullo stato attuale della teoria delle equazioni a derivate parziali.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 103–110. Edizioni Cremonese, Roma, 1955. 3000 Lire.

Egorov, V. G. Stability of solutions of systems of total differential equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 677–680. (Russian)

Let (1) $dx=P(u,x)du+Q(v,x)dv$, u, v scalars, x, P, Q n-vectors; the integrability condition $(\partial P/\partial x)\cdot Q=(\partial Q/\partial x)\cdot P$ is assumed, $P(u,0)=Q(v,0)=0$. The solution $x=0$ of (1) is stable if, given $\varepsilon>0$, there is a $\delta>0$ such that $\|x(u_0,v_0)\|<\delta$ implies $\|x(u,v)\|<\varepsilon$ for $u\geqq u_0$, $v\geqq v_0$; it is asymptotically stable if moreover $x(u_n,v_n)\to 0$ when $u_n\to\infty$, $v_n\to\infty$. Theorem 1: If the solutions $x=0$ of the (ordinary) systems (2) $dx=P(u,x)du$ and (3) $dx=Q(v,x)dv$ are stable, the solution $x=0$ of (1) is stable. Theorem 2: If moreover the solution $x=0$ of (3) is asymptotically stable, so is the solution of (1). Theorem 4 is an application to stability of ordinary differential equations of the fact that if A, B are constant permutable matrices whose characteristic roots $\lambda^{(a)}$, $\lambda^{(b)}$ satisfy $\mathrm{Re}\,\lambda^{(a)}<0$, $\mathrm{Re}\,\lambda^{(b)}\leqq 0$, then the characteristic roots of $A+B$ have negative real parts; the author derives this from his previous theorems, but the fact can be proved independently in several ways. Theorem 7: If $X(x)$, $Y(x)$ are homogeneous vector functions of degree $m\geqq 1$ and satisfy $(\partial X/\partial x)\cdot Y=(\partial Y/\partial x)\cdot X$, and if the solutions $x=0$ of the (ordinary) systems $\dot{x}=X$, $\dot{x}=Y$ are asymptotically stable, there is a number $L>0$ such that the solution $x=0$ of the (ordinary) system $\dot{x}=\alpha X+\beta Y+\alpha\Phi(\alpha t,x)+\beta\Psi(\beta t,x)$ is asymptotically stable for any positive constants α, β and

$$\|\alpha\Phi+\beta\Psi\|\leqq L\|x\|^m.$$

J. L. Massera (Montevideo).

Yanenko, N. N. Reduction of a system of quasilinear equations to a quasilinear equation. Uspehi Mat. Nauk (N.S.) **10**, no. 3(65) (1955), 173–178. (Russian)

Verfasser betrachtet Systeme von drei quasilinearen partiellen homogenen Differentialgleichungen in drei abhängigen Funktionen $u(x,y)$, $v(x,y)$, $w(x,y)$. Aus jeder Lösung, deren Gleichungen sich nach x, y auflösen lassen, ergibt sich eine Funktionalbeziehung $w=w(u,v)$. Kann die Funktion $w(u,v)$ durch eine Differentialgleichung charakterisiert werden? Eine solche gewinnt Verfasser in Form einer partiellen Differentialgleichung dritter Ordnung. Umgekehrt bestimmt sich eine Lösung $u=u(x,y)$, $v=v(x,y)$, $w=w(x,y)$ des quasilinearen Systems aus der Kenntnis eines Integrals $w(u,v)$ der erwähnten partiellen Differentialgleichung dritter Ordnung bis auf lineare Transformationen

$$x=cx'+c_1,\quad y=cy'+c_2$$

der unabhängigen Veränderlichen. Da die $w(u,v)$ bestimmende Differentialgleichung als Integrabilitätsbedingung eines Systems dreier linearer partieller Differentialgleichungen der Gestalt

$$A_i\frac{\partial x}{\partial u}+B_i\frac{\partial x}{\partial v}+C_i\frac{\partial y}{\partial u}+D_i\frac{\partial y}{\partial v}=0\quad(i=1,2,3)$$

erscheint, ist der Rang h dieses Systems wesentlich. Für $h=2$ ergeben sich zwei Bedingungen, aus welchen man im allgemeinen p und q als Funktionen von u und v bestimmen kann. Aus $dw=pdu+qdv$ bestimmt sich dann $w(u,v)$ bis auf eine additive Konstante, wofern $\partial q/\partial u=\partial p/\partial v$. Als Beispiel behandelt Verfasser das System aerodynamischer Differentialgleichungen

$$\frac{\partial v}{\partial t}+\frac{\partial p}{\partial x}=0,\quad \frac{\partial v}{\partial x}+bp^{-k}\frac{\partial p}{\partial t}=0,\quad \frac{\partial b}{\partial t}=0$$

im Falle lokal adiabatischer Gasströmungen (t Zeit, x Lagrangesche Koordinate, p Druck, v Geschwindigkeit, b Entropie) und vergleicht seine Ergebnisse mit den Resultaten einer Untersuchung von M. H. Martin [Canad. J. Math. **5** (1953), 37–39; MR **14**, 815]. Während M. H. Martin auf eine Monge-Ampèresche Differentialgleichung geführt wird, kommt Verfasser auf eine quasilineare Differentialgleichung. Auch kann Verfasser seine Methode bei vorgegebener Entropiefunktion anwenden. Der Druck erscheint dabei als Funktion der Entropie. Schließlich wird das Cauchysche Anfangswertproblem für die erwähnten aerodynamischen Gleichungen behandelt.

M. Pinl (Köln).

Gel'fand, I. M., and Šapiro, Z. Ya. Homogeneous functions and their extensions. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 3–70. (Russian)

Cet article utilise la théorie des distributions de L. Schwartz [t. I et II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833]. Les résultats essentiels sont relatifs à des parties finies d'intégrales divergentes [cf. Hadamard, Le problème de Cauchy, Hermann, Paris, 1930; Bureau, notamment Comm. Pure Appl. Math. **8** (1955), 143–202; MR **16**, 826]; les résultats sont obtenus par la méthode de M. Riesz de prolongement analytique [Acta. Math. **81** (1949), 1–223; MR **10**, 713].

§ 1. Soit $(x_+^\lambda)_{\varepsilon<0}$ la fonction $=x^\lambda$ si $x>0$, $=0$ si $x<0$, $\mathrm{Re}\,\lambda>-1$. L'application $\lambda\to(x_+^\lambda)_{\varepsilon>0}$ du demi plan $\mathrm{Re}\,\lambda>-1$ dans l'espace $\mathcal{D}'$ des distributions sur R, est holomorphe et se prolonge analytiquement au plan entier en une fonction méromorphe, avec des poles simples aux points -1, -2, $\cdots$. Le résidu au point $-k$ est

$$(-1)^{k-1}/(k-1)!\,\delta^{(k-1)}$$

($\delta^{(p)}$=dérivée d'ordre p de la masse de Dirac δ). Résultats analogues pour $(|x|^\lambda)_{\varepsilon<0}$, $|x|^\lambda$, etc.

§ 2. Soit dans R^n une fonction continue F, homogène de degré 1, >0 en dehors de l'origine (exemple: $|x|=r$); pour $\mathrm{Re}\,\lambda>-n$, $x\to F^\lambda(x)$ définit une distribution F_λ; $\lambda\to F_\lambda$ est holomorphe dans $\mathrm{Re}\,\lambda>-n$, à valeurs dans $\mathcal{D}'$; elle se prolonge analytiquement en une fonction méromorphe avec poles simples aux points $-n$, $-n-1$, $\cdots$. Etude des distributions homogènes de degré λ (i.e. $\langle T_\alpha,\varphi(x/\alpha)\rangle=\alpha^{\lambda+n}\langle T,\varphi\rangle$ pour toute $\varphi\in\mathcal{D}$=espace des fonctions indéfiniment différentiables à support compact). Exemple: soit Ω_n l'aire de la sphère unité dans R^n, le pole au point $-n$ de $2r^\lambda/\Omega_n\Gamma((\lambda+n)/2)$ est δ.

§ 3. Applications. a) Décomposition de δ. On désigne par Ω la sphère unité, $\omega=(\omega_1,\cdots,\omega_n)\in\Omega$, $d\omega$=élément d'aire superficielle. Si $f\in\mathcal{D}(R)$, soit f^* définie dans R^n par

$$f^*(x)=f(\omega_1x_1+\cdots+\omega_nx_n);$$

l'application $f\to f^*$ se prolonge par continuité en une application $T\to T^*$ de $\mathcal{D}'(R)$ dans $\mathcal{D}'(R^n)$. On désigne par $\delta^{(p)}(\omega_1x_1+\cdots+\omega_nx_n)$ l'image de $\delta^{(p)}$ dans cette application. Si n est impair on a:

$$\delta=\frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}}\int_\Omega\delta^{(n-1)}(\omega_1x_1+\cdots+\omega_nx_n)\,d\omega,$$

formule due à A. A. Hačaturov [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 3(61), 205–212; MR **16**, 229]. Formule analogue si n est impair. Ces formules découlent facilement des §§ précédents et de la formule

$$(*)\qquad \frac{1}{\pi^{(n-1)/2}\Gamma((\lambda+1)/2)}\int_\Omega|\omega_1x_1+\cdots+\omega_nx_n|^\lambda d\omega=2r^\lambda\Gamma((\lambda+n)/2).$$

b) Solutions élémentaires d'opérateurs différentiels el-

liptiques L à coefficients constants. On utilise (*); on cherche v_ω ne dépendant que de $\omega_1x_1+\cdots+\omega_nx_n$, solution de

$$Lv_\omega=|\omega_1x_1+\cdots+\omega_nx_n|^\lambda/\pi^{(n-1)/2}\Gamma((\lambda+1)/2)$$

(donc problème à une variable). On fait ensuite tendre λ vers $-n$.

c) Problèmes de Cauchy. Par des méthodes analogues on se ramène à des problèmes à deux variables. Dans le cas où l'opérateur différentiel est homogène, les A. retrouvent les formules de Herglotz-Petrowsky [pour une étude plus générale, cf. Leray, Hyperbolic differential equations, Inst. Advanced Study, Princeton, 1953; MR **16**, 139].

§ 4. Une fonction g définie sur R^n ou sur une sphère est dite équivalente à une fonction homogène au voisinage d'un point s'il existe un système de coordonnées locales dans lequel g est homogène. Soit G une fonction indéfiniment différentiable sur R^n. On définit la notion de point réductible M de la variété $G=0$ par récurrence sur la dimension; le point M est dit réductible si 1) G est équivalente à une fonction homogène au voisinage de M, 2) l'intersection de $G=0$ avec une sphère assez petite de centre M est composée de points réductibles sur la sphère (si $n=1$, on garde seulement 1)). Si l'on peut choisir le système de coordonnées locales de façon que G soit homogène de degré m et ne dépende que de k variables, M est dit d'ordre k et de degré m. On considère la distribution G^λ définie par $\int_{G>0}G^\lambda(x)\varphi(x)dx$, Re λ assez grand. Si la variété $G=0$ est bornée, formée de points réductibles d'ordre 1, l'application $\lambda\to G^\lambda$ se prolonge en une fonction méromorphe. A chaque composante connexe de $G=0$ composée de points de degré m correspond la suite de poles simples $-1/m$, $-2/m$, $\cdots$. Pour le calcul des résidus les A. utilisent un résultat de Leray [C.R. Acad. Sci. Paris **234** (1952), 1112–1115; MR **14**, 477]. Dans le cas général, les composantes de $G=0$ formées de points d'ordre r et de degré m donnent des poles (multiples) aux points $-r/m$, $-(r+1)/m$, $\cdots$.

§ 5. Transformation de Fourier de fonctions et distributions homogènes. On se ramène à une intégrale sur Ω par passage en coordonnées polaires. Tableau de formules.

J. L. Lions (Nancy).

Kiro, S. N. On analytic solutions of the equation

$$\frac{\partial^2u}{\partial x_1\partial x_2}=f\Big(x_1,\cdots,x_n;u;\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n},\frac{\partial^2u}{\partial x_1^2},\cdots,\frac{\partial^2u}{\partial x_n^2},\frac{\partial^2u}{\partial x_1\partial x_2},\cdots,\frac{\partial^2u}{\partial x_{n-1}\partial x_n}\Big).$$

Ukrain. Mat. Ž. **7** (1955), 29–46. (Russian)

It is shown that the equation in question has a real analytic solution taking given consistent real analytic values when x_1 or x_2 vanish, provided the right side is real and analytic and

$$4(\partial f/\partial u_{11})(\partial f/\partial u_{22})-1\leqq 0\quad(u_{11}=\partial^2u/\partial x_1^2,\ u_{22}=\partial^2u/\partial x_2^2)$$

when $x_1=x_2=0$. This result generalizes one by Günther [Mat. Sb. **32** (1924), 26–42].

L. Gårding (Lund).

Levitan, B. M. On the asymptotic behavior of a spectral function and expansion in eigenfunctions of the equation

$$\Delta u+\{\lambda-q(x_1,x_2,x_3)\}u=0.$$

Trudy Moskov. Mat. Obšč. **4** (1955), 237–290. (Russian)

This paper contains the proofs of results previously announced by the author [Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 179–182; MR **15**, 797]. They concern the asymptotic behavior of the spectral function for the problem $-\Delta u+qu=\lambda u$, $\partial u/\partial n=0$ on the boundary. An important device used is the relationship between the solutions of the auxiliary Cauchy problem $\Delta u-qu=u_{tt}$, $u(t=0)=f$, $u_t(t=0)=0$, and the solutions of the original boundary-value problem. [For a study of the same problems for general elliptic self-adjoint semi-bounded differential operators see L. Gårding, Kungl. Fysiogr. Sällsk. i Lund Förh. **24**, no. 21 (1955); MR **17**, 158.]

E. A. Coddington (Copenhagen).

Kay, I., and Moses, H. E. The determination of the scattering potential from the spectral measure function. I. Continuous spectrum. Nuovo Cimento (10) **2** (1955), 917–961.

This paper appeared earlier as a report [Div. Electromag. Res., Inst. Math. Sci. New York Univ., Res. Rep. No. **CX–18** (1955); MR **16**, 929].

Payne, Laurent E., Pólya, Georges, et Weinberger, Hans F. Sur le quotient de deux fréquences propres consécutives. C. R. Acad. Sci. Paris **241** (1955), 917–919.

Let $\{\lambda_n\}$ denote the ordered eigenvalues of the problem: $\Delta u+\lambda u=0$ in a domain, $u=0$ on the boundary. By an ingenious elementary argument the authors show that $\lambda_{n+1}\leqq 3\lambda_n$. It is conjectured that 3 can be replaced by 2.539 which is the correct value in case of a circle. Similarly for the problems $\Delta\Delta u-\lambda u=0$ and $\Delta\Delta u+\lambda\Delta u=0$ (in both cases with the boundary condition $u=u_\nu=0$) the inequalities $\lambda_{n+1}\leqq 9\lambda_n$ and $\lambda_2<3\lambda_1$ hold, respectively.

G. Szegö (Stanford, Calif.).

Pólya, G. On the characteristic frequencies of a symmetric membrane. Math. Z. **63** (1955), 331–337.

The membrane problem $u_{xx}+u_{yy}+\nu^2u=0$ is considered on a two-dimensional domain D. For the boundary condition $u=0$, the eigenvalues (values of ν) are called $\lambda_1<\lambda_2\leqq\lambda_3\leqq\cdots$, while for $\partial u/\partial n=0$ they are $0=\mu_1<\mu_2\leqq\mu_3\leqq\cdots$. The number r' (maximum inner conformal radius) is defined as the radius of the largest circle that can be mapped conformally onto D with unit linear magnification at the center. It is known [Pólya and Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951, pp. 97–98; Szegö, J. Rational Mech. Anal. **3** (1954), 343–356; MR **13**, 270; **15**, 877] that the quantities λ_1r' and μ_2r' attain their maxima when D is a circle.

D is said to possess symmetry of order n, if it is invariant under a rotation through an angle $2\pi/n$ about a point O. The following theorem is proved. There are two increasing functions $l(n)$ and $m(n)$ of the positive integer n such that if D is convex and possesses symmetry of order at least $2n+1$ but is not a circle, the quantities

$$\lambda_1r',\ \lambda_2r',\ \cdots,\ \lambda_{l(n)}r',\ \mu_2r',\ \cdots,\ \mu_{m(n)}r'$$

are less than the corresponding quantities for a circle.

The theorem is proved by transplantation. It might be noted that if r' is redefined as the radius of the circle that is mapped conformally onto D with unit linear magnification at its center and with its center mapped on the center of symmetry of D, the theorem holds when D is only simply connected rather than convex. It is also shown that for all simply connected D, the quantities λ_nr' are bounded by a function of n only.

H. F. Weinberger (College Park, Md.).

Forsythe, George E. Asymptotic lower bounds for the fundamental frequency of convex membranes. Pacific J. Math. **5** (1955), 691–702.

Let R be a simply connected open region of the (x, y)-plane whose closed boundary curve C is piecewise analytic and has convex corners. Let λ be the least eigenvalue of

$$(*) \qquad u_{xx}+u_{yy}=-\lambda u$$

under the condition $u=0$ on C. The paper deals with the relation of λ to the corresponding least eigenvalue λ_h of a suitably chosen finite-difference operator Δ_h over a network with small mesh width h. The network consists of the lines $x=\mu h$, $y=\nu h$ ($\mu, \nu=0, \pm1, \pm2, \cdots$). The points $(\mu h, \nu h)$ in R form the set R_h of interior nodes of the net. The set C_h of boundary nodes consists of all points $(\mu h, \nu h)$ on C and all isolated points of intersection of the net with C.

If (x, y) is a point of R_h, the neighbouring nodes are $(x-h_1, y)$, $(x+h_2, y)$, $(x, y-h_3)$, $(x, y+h_4)$, where $0<h_i\leqq h$; some of these neighbouring nodes may belong to C_h. Let v be any net function defined on the set of all nodes, interior and boundary, and let v vanish on C_h. Then define

$$\Delta^{(h)}v(x, y)=\frac{2}{h_1+h_2}\left\{\frac{1}{h_1}v(x-h_1, y)+\frac{1}{h_2}v(x+h_2, y)\right\}$$
$$+\frac{2}{h_3+h_4}\left\{\frac{1}{h_3}v(x, y-h_3)+\frac{1}{h_4}v(x, y+h_4)\right\}$$
$$-\left\{\frac{2}{h_1h_2}+\frac{2}{h_3h_4}\right\}v(x, y).$$

This reduces to the operator usually employed when all five nodes are within C. The operator Δ^h is not self-adjoint in that its matrix $A^{(h)}$ is not symmetric. Let A_h be the symmetric part of $A^{(h)}$, and let the transformation from v to $\Delta_h v$ be the transformation with matrix A_h. If λ_h is the least eigenvalue of

$$\Delta_h v=-\lambda_h v \text{ in } R_h$$

with the condition $v=0$ on C_h, it is shown that, as $h\to0$,

$$\frac{\lambda_h}{\lambda}\leqq1-ah^2+o(h^2).$$

The quantity a, which depends on the eigensolution u of (*), can probably be negative for certain non-convex R. But if R is convex, $a>0$; and then these exists a positive number h_0 such that $\lambda_h<\lambda$ for all $h<h_0$.

The author writes that Lemma 1, while true, is proved incompletely and (in part) incorrectly. *E. T. Copson.*

★**Synge, J. L. A technique for the solution of the biharmonic equation.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 39–53. Edizioni Cremonese, Roma, 1955. 3000 Lire.

The application of the method of the hypercircle [W. Prager and J. L. Synge, Quart. Appl. Math. **5** (1947), 241–269; MR **10**, 81] (or Schwarz's inequality [J. B. Diaz, Collect. Math. **4** (1951), no. 2, 3–49; MR **14**, 1084]) to a two-dimensional biharmonic problem arising in hydrodynamics is considered. The function ψ satisfies $\Delta\Delta\psi=0$ in a domain with boundary B. The value of ψ is given on B. In addition, $\partial u/\partial n$ is given on part B_1 of B, while the quantity $M(\psi)\equiv(\psi_{xx}-\psi_{yy})(n_2^2-n_1^2)-\psi\psi_{xy}n_1n_2$ is given on the remainder B_2 of B. Subscripts on ψ denote partial derivatives, while n_1 and n_2 are the components of the unit outward normal.

The procedure given in the paper may be interpreted as follows. If u is any function for which the above boundary operators vanish, then

$$\iint u\Delta\Delta u=\iint(u_{xx}-u_{yy})^2+4u_{xy}^2.$$

It is to the positive definite quadratic functional on the right that the variational technique must be applied. Suppose for simplicity that $M(\psi)=0$ on B_2. Then, if $v(x)$ satisfies the boundary conditions for ψ and $\partial\psi/\partial n$ on B and B_1 respectively and is twice piecewise continuously differentiable, and if p_1 and p_2 are twice piecewise continuously differentiable functions satisfying $p_{1xx}-p_{1yy}+2p_{2xy}=0$ together with $p_1(n_2^2-n_1^2)-2p_2n_1n_2=0$ on B_2, we have by the divergence theorem and Schwarz's inequality

$$\left[\iint p_1^2+p_2^2\right]^{-1}E(\psi, p)^2\leqq\iint(\psi_{xx}-\psi_{yy})^2+4\psi_{xy}^2$$
$$\leqq\iint(v_{xx}-v_{yy})^2+4v_{xy}^2,$$

where

$$-E(\psi, p)\equiv\oint_B\psi\Big[p_{1x}n_1-p_{1y}n_2+p_{2x}n_2+p_{2y}n_1$$
$$-\frac{\partial}{\partial s}\{p_2(n_2^2-n_1^2)+2p_1n_1n_2\}\Big]ds$$
$$+\int_{B_2}\frac{\partial\psi}{\partial n}[p_1(n_2^2-n_1^2)-2p_2n_1n_2]ds.$$

This integral can be evaluated in terms of the given boundary quantities for ψ. Equality occurs in one case for $p_1=\psi_{xx}-\psi_{yy}$, $p_2=2\psi_{xy}$, and in the other case for $v=\psi$. A careful examination of the divergence theorem shows that the inequality on the left still holds if p_1 and p_2 are permitted to have certain related jump discontinuities across a finite number of arcs.

Unfortunately, the simplicity and universality of the variational method is obscured in the paper by the introduction of extraneous notation inspired by the hydrodynamical significance of the problem. The two-vector (p_1, p_2) is replaced by a 2×2 matrix subject to two constraints (strain matrix with incompressibility) and this, in turn, is represented in terms of another matrix (stress). The single function v is represented by its derivatives $-v_y$ and v_x (velocity).

The chief contribution of the paper is to construct two sets of trial functions admissible in the two variational principles, satisfying homogeneous boundary conditions, vanishing outside an arbitrarily small region, and so simple that scalar products may be computed with relative ease. These may be used to obtain arbitrarily close approximations by a Rayleigh-Ritz procedure. The functions are of two kinds (one for each bound). They are the analogues of the pyramid functions used by the author in the approximation of certain harmonic problems [Rend. Mat. e Appl. (5) **10** (1951), 24–44; MR **16**, 257]. Both kinds of functions vanish outside a triangulated octagon. The function v of the first kind is piecewise quadratic with discontinuities in the second derivatives across certain of the triangulating lines. The function of the second kind is a vector function (p_1, p_2). It is piecewise constant with admissible discontinuities across some of the triangulating lines. *H. F. Weinberger.*

★**Courant, R. Remarks and problems concerning hyperbolic systems.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 168–173. Edizioni Cremonese, Roma, 1955. 3000 Lire.

This paper first sketches the reduction of the definition

of hyperbolicity for linear problems with $n+1$ independent variables to that for two variables, which depends, in turn, on the simplicity of the elementary divisors of a certain matrix. Next, an averaging method is indicated for reducing a problem with constant coefficients in $n+1$ variables to a problem in two variables. Then a numerical procedure is described in the case of two variables in which variable coefficients are approximated locally by constants, and the characteristics of the approximating equations are used to set up a simple recursive scheme for stepping from $t=N\tau$ to $t=(N+1)\tau$. Finally, it is pointed out that both theoretical and experimental study is needed of the similar numerical method for more than two variables. *F. A. Ficken* (Knoxville, Tenn.).

Durand, Emile. Identités conduisant aux solutions des équations aux dérivées partielles linéaires et à coefficients constants. Bull. Soc. Math. France **82** (1954), 361–411.

This paper is concerned chiefly with second-order equations. Starting with the expression for the Laplacian of a volume potential and after elementary operations which include interpreting one of the independent variables as imaginary, the author derives identities for the solutions of hyperbolic equations with constant coefficients. These identities give the value of the dependent variable at a point as an expression which is obtained by applying a differential operator L to an integral I. They are meaningful in the ordinary way, and they are of the same form in spaces of even and of odd numbers of dimensions. The integral I in the identities, which contain the dependent variable, are extended, not over lower-dimensional initial manifolds, but over domains which have the same number of dimensions as there are independent variables. However, the identities lead to formulas for the solution of Cauchy's and of Kirchhoff's problems when the differentiations contained in L are carried out. *A. Douglis* (New York, N.Y.).

Vasilache, Sergiu. Sur une nouvelle équation des télégraphistes. Rev. Math. Phys. **2** (1954), 33–50 (1955). Translation of Acad. R. P. Române. Stud. Cerc. Mat. **3** (1952), 295–320; MR **16**, 484.

Herriot, John G. The solution of Cauchy's problem for a third-order linear hyperbolic differential equation by means of Riesz integrals. Pacific J. Math. **5** (1955), 745–763.

Dans R^3, l'auteur considère l'équation $\partial^3 u/\partial x_1\partial x_2\partial x_3 = h(x)$; il donne $u(x)$ et ses dérivées jusqu'à l'ordre deux, inclusivement, sur le plan d'équation $x_1+x_2+x_3=0$ et résout le problème de Cauchy dans le demi-espace Ω d'équation $x_1+x_2+x_3>0$, les données de Cauchy et le second membre $h(x)$ de l'équation étant supposés suffisamment réguliers. La solution dont les données de Cauchy sont nulles s'écrit

$$I^1h=\int_{D(x)} h(\xi)d\xi,$$

où $D(x)$ désigne la partie de Ω comprise dans un trièdre trirectangle d'origine x, opposé au trièdre de référence. L'auteur établit ce résultat en étudiant la généralization suivante de l'intégrale de Riemann-Liouville

$$I^p h=\frac{1}{[\Gamma(p)]^3}\int_{D(x)} h(\xi)\Big[\prod_{i=1}^{3}(x_i-\xi_i)\Big]^{p-1}d\xi \quad (\Re\, p>0),$$

pour laquelle

$$I^pI^qh=I^{p+q}h,\ (\partial^3/\partial x_1\partial x_2\partial x_3)I^{p+1}h=I^ph,\ \lim_{p\to 0} I^ph=h$$

[cf. M. Riesz, Acta Math. **81** (1949), 1–223; MR **10**, 713]. L'auteur étudie d'une manière analogue la solution relative au problème général. Il fait la synthèse des formules résolutives qu'il obtient. *H. G. Garnir* (Liège).

Vodička, Václav. Hollow circular cylinder under periodic fluctuations of temperature. Appl. Sci. Res. A. **5** (1955), 327–337.

A formula is derived for steady periodic temperatures $u(r, z, t)$ at points within a cylindrical shell when linear surface heat transfer takes place from the bases and from the cylindrical surfaces into media whose temperatures vary periodically with time. The boundary condition written for each surface is of the type $du/dn=(u-Be^{i\omega t})h$, where n denotes distance normal to the surface and ω, B and h may each have distinct constant values on the four surfaces. Separation of variables and superposition of solutions are used to write the formula in terms of infinite series. Although the periodic variation of temperatures at points within the solid should differ in phase from the variation of the temperatures of the bordering media, the author's procedure of using imaginary time functions fails to display such phase differences. *R. V. Churchill.*

Vodička, Václav. Eindimensionale Wärmeleitung in geschichteten Körpern. Math. Nachr. **14** (1955), 47–55.

Let $u_i(x, t)$ $(i=1, 2, \cdots, n)$ denote the solutions of this boundary-value problem: $\partial u_i/\partial t=a_i^2(\partial^2 u_i/\partial x^2+cx^{-1}\partial u_i/\partial x)$ $(x_i<x<x_{i+1},\ t>0)$; $\partial u_1/\partial x=h_0'u_1$ $(x=x_1)$, $\partial u_i/\partial x=h_i(u_{i+1}-u_i)$ and $\partial u_{i+1}/\partial x=h_i'(u_{i+1}-u_i)$ when $x=x_{i+1}$ $(i=1, 2, \cdots, n-1)$; $\partial u/\partial x=-h_nu_n$ $(x=x_{n+1})$; $u_i(x, 0)=f_i(x)$ $(x_i<x<x_{i+1})$. Here a_i, h_i and h_i' denote positive constants and the x_i's are constants such that $0\leq x_1<x_2<\cdots<x_{n+1}$. A general formula for $u_i(x, t)$ is described with the aid of separation of variables and formal application of the modified orthogonality relations between the resulting characteristic functions of the variable x. When $c=0$, $u_i(x, t)$ represents temperatures in a composite slab with linear surface heat transfer into media at temperature zero at the outer faces $x=x_1$, $x=x_{n+1}$, and with an arbitrary initial temperature distribution $f_i(x)$. When $n=1$, the problem is the corresponding one for a composite cylindrical shell, where x represents distance from the axis of the shell. When $n=2$ the body is a composite spherical shell and x represents distance from the center. In all cases the above interface conditions at $x=x_{i+1}$ $(i=1, 2, \cdots, n-1)$ represent linear surface heat-transfer conditions, rather than the more common interface conditions. *R. V. Churchill* (Ann Arbor, Mich.).

★ **Mura, Toshio, and Kinosita, Nobuo. Stefan like problem of a cylinder.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 345–348. Science Council of Japan, Tokyo, 1955.

The problem considered is a condensation-time problem of a cast iron cylinder. In terms of dimensionless quantities, this is the problem of finding a function $\theta(x, \tau)$ which satisfies the heat equation

$$1)\ \theta_{RR}+\frac{1}{R}\theta_R=\theta_\tau,\ \bar{R}(\tau)<R\leq 1$$

subject to the boundary conditions:

$$2)\ \theta=0,\ \text{at } R=1,\ 3)\ \theta=1,\ \text{at } R=\bar{R}(\tau),$$
$$4)\ \theta_R=\mu\dot{\bar{R}},\ \text{at } R=\bar{R}(\tau),$$

and the initial conditions:

5) $\bar{R}(\tau)=1$, at $\tau=0$, 6) $\theta=1$, at $\tau=0$.

The authors use the method of Laplace transformation to find a solution of this problem which is valid in the neighborhood of the surface of the cylinder. A solution by conformal mapping which is valid near the center of the cylinder is used to study the situation there. Both solutions are exhibited graphically as well as experimental data which falls roughly in between these two solution curves. *C. G. Maple* (Ames, Ia.).

★ **Page, Chester H. Physical mathematics.** D. Van Nostrand Company, Inc., New York, 1955. x+329 pp. $7.50.

The author's aim is "to explain some of the classical mathematical techniques from the viewpoint of a physicist, using physical arguments freely to indicate propositions to be tested." The mathematics treated here is limited to analysis. The book touches upon a substantial number of those parts of analysis that are associated with physics. Plausible reasons for conclusions, rather than mathematically sound arguments and full conditions for the validity of results, are stressed. This program lends itself to short derivations and to several interesting innovations in derivations. The reader is expected to be acquainted with various mathematical concepts beyond the calculus. Impulse functions, for instance, are used in several derivations but not explicitly defined. The book contains some illustrative examples, but no sets of exercises for the reader to solve. The coverage is indicated by the chapter headings: Integration and differentiation, The vibrating string, The general problem in three dimensions, Eigenfunctions and eigenvalues, Geometric interpretation of eigenfunctions, The separated equations, The nonhomogeneous problem, Integral equations, Methods of solving integral equations, The unbounded region, The Fourier integral, Passive physical systems, Viewpoints on Fourier transforms, The Laplace transform, Transient analysis, Difference equations and partial differential equations. *R. V. Churchill.*

See also: Jaeger, p. 343; Marinescu, p. 386; Gel'fand and Kostyučenko, p. 388; Crandall, p. 413; Abramowitz and Cahill, p. 413.

Difference Equations, Special Functional Equations

Kamynin, L. I. On the behavior of a solution of a finite-difference analogue of the wave equation. Prikl. Mat. Meh. **19** (1955), 589–598. (Russian)

The equation considered is

$$u_{m,n+1}=a^2(u_{m+1,n}+u_{m-1,n})+2(1-a^2)u_{m,n}-u_{m,n-1},$$

where $m=0, \pm1, \pm2, \cdots, n=0, 1, 2, \cdots$, and $0<a^2<1$. The solution $u_{m,n}$ satisfying the initial conditions $u_{m,0}=0$, $u_{m,1}=0$ if $m\neq0$, $u_{0,1}=1$, is a polynomial in a whose coefficients are determined. In particular, $u_{m,n}=0$ if $|m|\geqq n$. It is also shown that $u_{qn,n}=O(n^{-2/3})$ for $a<q<1$, $u_{qn,n}=(6a)^{-1}+O(n^{-2/5})$ for $q=a$, $u_{qn,n}=(2a)^{-1}+O(n^{-2/5})$ for $0\leqq q<a$, and $u_{m,n}=(2a)^{-1}+O(n^{-2/3})$ for m fixed. *N. D. Kazarinoff* (Lafayette, Ind.).

van den Berg, Jw. Über die Funktionalgleichung $\phi(\alpha x)-\beta\phi(x)=F(x)$. I, II. Nieuw Arch. Wisk. (3) **3** (1955), 79–88, 113–123.

This is a clearly written, thorough and detailed study of the functional equation figuring in the title for real variables and arbitrary real α, β. The author's aim is to give the general real, resp. bounded, solutions and the conditions which must be imposed on $F(x)$ in order to secure the existence of such solutions. His point of departure is a paper of H. Hadwiger [Elem. Math. **5** (1950), 86–88; MR **12**, 416]. Both authors seem to be unaware of former researches concerning these equations, such as D. Pompeiu, Bull. Math. Phys. Pures Appl. Ecole Polytech. Bucarest **9** (1938), 54–56; R. Badescu, Atti 2° Congresso Un. Mat. Ital., Bologna, 1940, Edizioni Cremonese, Rome, 1942, pp. 154–161 [MR **8**, 385] and of the fact that the functional equation in question can be reduced to difference equations. E.g., for $\alpha>0$, $x>0$ by the substitution $x=\alpha^u$, $\phi(\alpha^u)=f(u)$, $F(\alpha^u)=g(u)$ reduces it to $f(u+1)-\beta f(u)=g(u)$ and this gives, e.g., for $\beta>0$ the general solution

$$\phi(x)=\beta^{\alpha\log x}\,\mathrm{S}\frac{F(\alpha^u)}{\beta^{u+1}}\Big|_{u=\alpha\log x}$$

[$\mathrm{S}G(u)=H(u)$ is the "sum" of $G(u)$: $H(u+1)-H(u)=G(u)$]. In particular, $\phi(x)=\beta^{\alpha\log x}\varphi^{(\alpha\log x-[\alpha\log x])}$ for the homogeneous equation [$F(x)=0$], which is equivalent to the solution (6) given by the author on p. 82. It might be pointed out that this formula

$$\phi(x)=\beta^{[\alpha\log|x|]}\omega\{|x|\alpha^{-[\alpha\log|x|]}\}\quad(x\neq0)$$

of the author gives a general solution of the homogeneous equation for every real $x\neq0$ only if the function $\omega(u)$ figuring in it is allowed to be different according as $x\lessgtr0$, which follows also from a remark of the author (p. 81, line 19). The paper ends with an examination of some unbounded solutions, of continuity and derivability, and of the equation $\phi(\alpha x)-\beta(\alpha)\phi(x)=0$ [H. W. Pexider, Monatsh. Math. Phys. **14** (1903), 293–301]. *J. Aczél.*

See also: Forsythe, p. 373; Vekua, p. 377; Slugin, p. 387; Garwick, p. 412.

Integral Equations

Krein, M. G. On a new method of solution of linear integral equations of first and second kinds. Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 413–416. (Russian)

Let $g(x, a)$ be the solution $\phi(x)$ of

$$(*)\qquad \int_0^a K(x, s)\phi(s)ds-\mu\phi(x)=f(x)$$

in the case $f(x)\equiv1$, where $K(x, s)$ is measurable and $\mu\neq0$ is not an eigen-value. Let $g^*(x, a)$ be the corresponding solution for the transpose of K, and write $M(a)=\int_0^a g(s,a)ds$ so that $M'(a)=-\mu g(a, a)g^*(a, a)$. Then the solution of (*) for general $f(x)\in C(0, a)$ is

$$\phi(x)=(M'(a))^{-1}[(d/da)\int_0^a g^*(s, a)f(s)ds]g(x, a)-$$

$$g(x, u)(d/du)((M'(u))^{-1}(d/du)\int_0^u g^*(s, u)f(s)ds)du,$$

the non-vanishing of $M'(u)$ being postulated. A similar rule is then found to solve

$$\int_{-a}^a H(|x-s|)\phi(s)ds-\mu\phi(x)=f(x),$$

an interesting example being

$$-\phi(x)+\tfrac{1}{2}\varrho\int_{-a}^{a}J_1(\varrho|x-s|)\phi(s)ds=f(x).$$

The rule may even extend to the case $\mu=0$, it being then required that $H(x)$ should have a singularity at $x=0$ of the form $\log x$ or x^{-h} $(0<h<1)$. A number of special cases are discussed. [Previous relevant papers of the author: same Dokl. (N.S.) **94** (1954), 987–990; **97** (1954), 21–24; MR **16**, 38, 372.] [Reviewer's remark: There appears to be an error in the author's resolvent formula (2).] *F. V. Atkinson* (Canberra).

Takesada, Tositaka. On the singular point of integral equations of Volterra type. J. Math. Soc. Japan **7** (1955), 123–136.

The author considers the system

$$(1)\qquad xu_j(x)=\sum_{i=1}^{n}\int_0^a K_{ji}(x,t)u_i(t)dt \quad (j=1,\cdots,n),$$

where the $K_{ji}(t)=\sum_{\alpha,\beta}a_{ji}^{\alpha\beta}x^\alpha t^\beta$ $(\alpha,\beta\geqq 0)$ are analytic for $|x|<r_1$, $|t|<r_2$. It is proved that in the neighborhood of the singular point $x=0$ (1) has convergent series solutions of the type analogous to those that occur in the Fuchsian theory of the regular singular point of ordinary linear differential equations. Of importance are the roots λ_k of the equation

$$\det(a_{i,j}^{0,0}-\lambda\delta_{i,j})=0\ (i,j=1,\cdots,n;\ \delta_{i,j}=0\ (i\neq j),\ \delta_{i,i}=1).$$

Case (I): λ_k is a simple root and $\lambda_j-\lambda_k$ $(j\neq k)$ is not an integer. Case (II): λ_k is a multiple root or for some integer m one has $\lambda_k+m=\lambda_j$ for some j $(\neq k)$. The author makes a detailed study of each case, first obtaining formal series solutions and then proving their convergence.

W. J. Trjitzinsky (Urbana, Ill.).

Hvedelidze, B. V. Some composition formulas for singular integrals and their applications to the inversion of a Cauchy-type integral. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 81–88. (Russian)

Let C be a finite collection of open Liapounoff curves, $C=\sum_1^m C_k$ in the plane ω; a_k, b_k are the initial and terminal points of C_k; the a_k, b_k are arranged as $c_1,\cdots,c_{2m}$; $D=\omega-C$; $R_a^2(z)=\prod(z-a_k)$, $R_b^2(z)=\prod(z-b_k)$, $R=R_aR_b$; $R(z)\sim+z^m$ for $z\to\infty$. A positive direction is assigned on C; $R(t)=R^-(t)$ on C. A class $L^p(C;\omega(t))$ consists of the $f(t)$, for which $\int_C\omega(t)|f^p(t)|\,|dt|<\infty$, $\omega(t)\geqq 0$ being measurable. A class $H^*(C)$ consists of the $f(t)$ of class (H) (Hölder) on every closed subset of C, such that near every end point c one has $(t-c)^\alpha f(t)\in(\mathrm{H})$ for $0\leqq\alpha<1$. Principal integrals over C are introduced thus:

$$\pi iS\varphi=\int\nu dt,\ \pi iR(t_0)U\varphi=\int R(t)\nu dt,\ \pi iV\varphi=R(t_0)\int R^{-1}(t)\nu dt,$$

$$\pi iW\varphi=\gamma(t_0)\int\gamma^{-1}(t)\nu dt,\ \pi iP\varphi=\int\gamma(t)\nu dt,\ \pi iQ\varphi=\int\gamma^{-1}(t)\nu dt;$$

here $\nu=\varphi(t)(t-t_0)^{-1}$, $\gamma(t)=R_a(t)R_b^{-1}(t)$ and t_0 is interior C. If $\varphi\in L^{[p]}=L^p(C;|R|^{p-1})$, $p>1$, then a.e. on C one has: $SU\varphi=\varphi$, $US\varphi=\varphi(t_0)+R^{-1}(t_0)Q_{m-1}(t_0)$ (Q_{m-1} a polynomial); similar results for VS, SV. In the class $L_{a,b}^{[p]}=L^p(C;|R_a|^{-1}|R_b|^{p-1})$ $(p>1)$ one has a.e. $WS\varphi=SW\varphi=\varphi$; analogous results for PQ, QP. Amongst applications to inversion problems are the following. If $f\in L^{[p]}$, all solutions of (1) $S\varphi=f$ in $L^{[p]}$ are of the form $Uf+R^{-1}P_{m-1}$ (P_{m-1} a polynomial); (1) is also effectively solved in $L^p(C;|R|^{-1})$, when f is in the same class. If $f\in L_{a,b}^{[p]}$ $(p>1)$, then $P\varphi=f$ has a unique solution $\varphi=Qf$ in $L_{a,b}^{[p]}$; the converse is true. *W. J. Trjitzinsky.*

Gegeliya, T. G. On boundary values of integrals of Cauchy type for unsmooth surfaces. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 481–488. (Russian)

The author studies in Euclidean three-space the integral

$$(1)\qquad \Phi(P)=\iint_S M(Q,P)\varphi(Q)r^{-2}dS_Q \quad (P\in S),$$

where $r=r(Q,P)$ is the distance between points P, Q; S is a rectifiable surface without self-intersections, $M=(m_{ij})$ $(i,j=1,\cdots,n)$ is a matrix, and $\varphi=(\varphi_1,\cdots,\varphi_n)$ is a vector on S. Under various conditions formulas of type (1) for the boundary values of integrals are of importance for the theory of singular integral equations [e.g., for surfaces which are not closed (that is, that may have edges) see a paper of the reviewer's Acta Math. **84** (1950), 1–128; MR **12**, 505; for closed surfaces or closed manifolds see, e.g., G. Giraud, Ann. Sci. Ecole Norm. Sup. (3) **51** (1934), 251–372; A. V. Bicadze, Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 525–538; MR **15**, 533]. The author establishes a number of boundary-values formulas under conditions which along certain directions are more general than those involved in earlier works. Thus, for example, S may possess certain suitably mild forms of unsmoothness and may be subject to conditions lighter than those of Liapounoff. In a later paper the author intends to make applications of the results obtained to a study of singular integral equations.

W. J. Trjitzinsky (Urbana, Ill.).

Steinberg, J. Sur une classe d'équations intégrales singulières. Technion. Israel Inst. Tech. Sci. Publ. **6** (1954/5), 85–93. (Hebrew summary)

The author studies the equation

$$(1)\qquad f(s)-\lambda\int_{-\infty}^{\infty}K(\mu s-t)f(t)dt=0.$$

Let μ not be a root of unity and let $K(u)$ (a) be analytic in $u=u'+iu''$ in a strip B containing the axis of reals, (b) $|K(u)|=o(|u'|^{-r})$ for every $r>0$, when $u\to\infty$ in B, (c) $1/\lambda_1=\int_{-\infty}^{\infty}K(u)du\neq 0$; then for μs in B (1) has characteristic values $\lambda_n=\lambda_1\mu^{1-n}$ $(n=1,2,\cdots)$, for each of which (1) has a polynomial solution of degree $n-1$. If $\mu=\exp\{2\pi iMN^{-1}\}$ $(M<N$ integers) and (a), (c) hold, while $|K(u)|=o(e^{-h|u'|})$ $(h>0$, for $u\to\infty$ in $B)$, then for μs in B there exist N domains containing the $\lambda_n=\lambda_1\mu^{1-n}$, respectively, such that every λ within these domains is a characteristic value of (1). *W. J. Trjitzinsky.*

Karcivadze, I. N. On a formula of interchange of integrals. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 3–10. (Russian)

The following is proved. With $H(x,t)$ possibly complex-valued and $q>0$, let $|H|^q$ be summable on S $(0\leqq x,t\leqq 2\pi)$; if $\varphi(x)\in L_p(0,2\pi)$ $(p^{-1}+q^{-1}=1)$, then for almost all x on $(0,2\pi)$:

$$(1)\qquad \int H(x,t)dt\int\varphi(s)\operatorname{ctg}\frac{t-s}{2}ds=\int\varphi(s)ds\int H(x,t)\operatorname{ctg}\frac{t-s}{2}dt,$$

$$(2)\qquad \int\operatorname{ctg}\frac{x-s}{2}ds\int H(s,t)\varphi(t)dt=\int\varphi(t)dt\int\operatorname{ctg}\frac{x-s}{2}H(s,t)ds,$$

each member being in $L_q(0,2\pi)$. If $\varphi\in L_p(0,2\pi)$ and the q/p-th power of $\int|H(x,t)|^p dt$ is summable on $(0,2\pi)$, then (1) and (2) will hold and the members in (1), (2) will belong to $L_p(0,2\pi)$. Analogous results are given when integrations are along a simple closed curve of continuous curvature. *W. J. Trjitzinsky* (Urbana, Ill.).

Vekua, N. On a boundary problem of linear relationship for several unknown functions with given displacements. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 169–189. (Georgian. Russian summary)

D^+ is a bounded domain in the plane of $z=x+iy$, whose frontier L is simple, closed, suitably smooth; D^+ is to the left of L, when L is described in the positive direction. Functions $\alpha_k(t)$ $(k=1, \cdots, n)$ are supposed to transform L uniformly into L, t and $\alpha_k(t)$ describing L in opposite directions; the $\alpha_k'(t)$ exist, are of class (H) (Hölder) and are nonzero on L. The author finds vectors $\varphi(z)$, $\psi(z)$, meromorphic in D^+, such that

$$(1)\quad \varphi_j^+[\alpha_j(t_0)]=\sum_{k=1}^{n} G_{j,k}(t_0)\psi_k^+(t_0)+g_j(t_0)\quad (j=1, \cdots, n),$$

where the $G_{j,k}$, g_j are assigned beforehand in (H) on L; $\det(G_{j,k}(t_0))\neq 0$ on L. A problem of T. Carleman [Verh. Internat. Math.-Kongresses, Zürich, 1932, Bd. I, Füssli, Zürich-Leipzig, 1932, pp. 138–151] is fully solved, as a consequence. The theory is based on the observation that (1) is solved in the form:

$$\psi_j=\frac{1}{2\pi i}\int_L \frac{\mu_j(t)dt}{t-z}+R_j(z),\ \varphi_j(z)=\frac{1}{2\pi i}\int_L \frac{v(t)}{t-z}dt+\gamma_j(z),$$
$$v(t)=\sum_k G_{j,k}[\beta_j(t)]\mu_k[\beta_j(t)]+g_j[\beta_j(t)],$$

where $R_j(z)$, $\gamma_j(z)$ are rational functions and μ is a vector to be found in (H) with the aid of the theory of the usual singular integral equations.

W. J. Trjitzinsky (Urbana, Ill.).

Bicadze, A. V. On two-dimensional integrals of Cauchy type. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 177–184. (Russian)

Let $\mathfrak{q}(P)\equiv(q_1, q_2, q_3, q_4)$ be a continuously derivable vector in a domain D in Euclidean three-space U; D^+ is a simply connected domain bounded by a Liapounoff surface S, D^++SCD; $D^-=U-(D^++S)$. Various matrices are introduced as follows. $D^*(X, Y, Z)$, $D(X, Y, Z)$ are matrices with rows $(0, X, Y, Z)$, $(X, 0, Z, -Y)$, $(Y, -Z, 0, X)$, $(Z, Y, -X, 0)$ and with rows $(0, X, Y, Z)$ $(X, 0, -Z, Y)$, $(Y, Z, 0, -X)$, $(Z, -Y, X, 0)$, respectively; $\varrho(P, Q)$ is the distance between points $P(x, y, z)$, $Q(\xi, \eta, \zeta)$;

$$M(P, Q)\equiv -D^*\Big(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\Big)\frac{1}{\varrho}\cdot D(\alpha, \beta, \gamma),$$
$$N(P, Q)\equiv -D^*\Big(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\Big)\frac{1}{\varrho}\cdot D\Big(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\Big).$$

If $\mathfrak{q}\in$ (H) (Hölder) on S, then the 'Cauchy' integral $\mathfrak{p}(P)=(4\pi)^{-1}\iint M(P, Q)\mathfrak{q}(Q)d\omega_Q$ (over S) is analytic in D^+, D^-; on S such integrals exist in the sense of principal values; there are analogues of the Plemelj-Sohockiĭ formulas (for boundary values) and the Poincaré-Bertrand formulas for exchange of order of principal integrals, Results of this kind naturally lead to an effective study of singular integral equations such as

$$A\varphi(P_0)+\frac{1}{2\pi}B\iint_S M(P_0, Q)\varphi(Q)d\omega_Q=\mathfrak{f}(P_0),\ P_0\in S,$$

where the matrices A, B and the vector f are assigned, of class (H), on S. *W. J. Trjitzinsky* (Urbana, Ill.).

Ko, Dya-Ha. Über die Existenz und Eindeutigkeit der Lösungen von Integro-Differentialgleichungen. J. Chinese Math. Soc. **2** (1953), 275–287. (Chinese. Russian summary)

Bekanntlich wendet man zum Beweis der Existenz einer Lösung eine Fixpunktmethode an, die von Banach begründet und von Caccioppoli mit Erfolg angewendet worden ist. In der vorliegenden Arbeit wird mittels dieser Methode die Existenz und Eindeutigkeit der Lösungen folgender Typen von Integrodifferentialgleichungen bewiesen:

$$\text{I.}\quad \frac{d^m y(x)}{dx^m}+\lambda\sum_{k=0}^{m-1}a_k(x)\frac{d^k y(x)}{dx^k}+\lambda\int_a^b\sum_{k=0}^{m-1}R_k(x, t)\frac{d^k y(t)}{dt^k}dt=f(x),$$

wo $a_k(x)$, $R_k(x, t)$, $f(x)$ gegebene in $[a, b]$ stetige Funktionen sind, λ ein Parameter, $m\geqq n$.

$$\text{II.}\quad \frac{d^m y}{dx^m}+\sum_{k=0}^{m-1}a_k(x)\frac{d^k y}{dx^k}+\int_a^x K\Big(x, t, y(t), \cdots, \frac{d^n y(t)}{dt^n}\Big)dt=0,$$

wo $a_k(x)$ in $[a, b]$ stetige Funktionen sind, und K eine stetige Funktion ihrer Argumente, die der Lipschitzbedingung genügt:

$$K\Big(x, t, y^{(1)}, \frac{dy^{(1)}}{dt}, \cdots, \frac{d^n y^{(1)}}{dt^n}\Big)$$
$$\Big|-K\Big(x, t, y^{(2)}, \frac{dy^{(2)}}{dt}, \cdots, \frac{d^n y^{(2)}}{dt^n}\Big)\Big|\leqq M\sum_{k=1}^{n}\Big|\frac{d^k y^{(1)}}{dt^k}-\frac{d^k y^{(2)}}{dt^k}\Big|.$$

$$\text{III.}\quad \frac{d^{m_i} y}{dx^{m_i}}$$
$$+R_i(x, y_1, y_1', \cdots, y_1^{(m_1+1)}, \cdots, y_n, y_n', \cdots, y_n^{(m_n-1)})$$
$$+\int_a^x K_i(x, t, y_1(t), \cdots, y^{(p_1)}(t), \cdots, y_n(t), \cdots, y^{(p_n)}(t))dt=0,$$

wo R_i und K_i $(i=1, \cdots, n)$ stetige Funktionen ihrer Argumente sind und einer Lipschitzbedingung genügen.

$$\text{IV.}\quad \frac{dy_i(x)}{dx}=\sum_{k=1}^{\infty}a_{ik}(x)y_k(x)+b_i(x)\quad (i=1, 2, \cdots),$$

wo $a_{ik}(x)$ und $b_i(x)$ auf $[a, b]$ stetig sind und $\sum_{k=1}^{\infty}|a_{ik}(x)|\leqq M<\infty$ und $|b_i(x)|\leqq N$ $(i=1, 2, \cdots)$ gilt.

Autoreferat (Zbl **51**, 82).

Busbridge, I. W. On the X- and Y-functions of S. Chandrasekhar. Astrophys. J. **122** (1955), 327–348.

The solutions, $X(u)$ and $Y(u)$ of the system of integral equations

$$X(u)=1+u\int_0^1\frac{\Psi(v)}{u+v}[X(u)X(v)-Y(u)Y(v)]dv,$$
$$Y(u)=e^{-\tau/u}+u\int_0^1\frac{\Psi(v)}{u-v}[X(v)Y(u)-X(u)Y(v)]dv,$$

which arise in problems connected with finite atmospheres of optical thickness τ, were investigated by several authors. Approximations to these functions were obtained, but the existence of the solutions of the integral equations has never been proved rigorously, except in the limiting cases $\tau=0, \infty$.

Assuming that $\Psi(u)$ is an even, positive, and integrable function such that $\int_0^1\Psi(u)\leqq\frac{1}{2}$, the author first obtains (singular) integral equations satisfied by each of the two functions $X(u)\pm Y(u)$. She then transforms these singular integral equations into integral equations of the Fredholm type, and proves that the Neumann series for the latter integral equations converge for sufficiently large τ, thus

proving the existence of $X(u)$ and $Y(u)$ for all sufficiently large τ. *A. Erdélyi* (Pasadena, Calif.).

Calculus of Variations

Tietze, Heinrich. Bemerkungen zu Carathéodory's Einführung in Euler's Arbeiten über Variationsrechnung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1955**, 1–7.

On pp. XIII–XV of Euler's Opera omnia, ser. 1, vol. 24 [Soc. Sci. Nat. Helv., Bern, 1952; MR **15**, 89] Carathéodory described in modern terminology the justification for the Bernoulli-Euler method for deriving the differential equations of a minimizing curve. This description is in terms of an approximation to the minimizing curve based on a partition of the interval of abscissas into a finite number of equal parts, in place of the partition of the curve into infinitely many infinitely small parts as described by Euler. The note by Tietze is concerned with a difficulty which arises at the ends of the interval of abscissas. At the end he remarks that it seems questionable that the same procedure will work all the way to secure the differential equation when the interval of abscissas is partitioned arbitrarily into a finite number of parts. *L. M. Graves* (Chicago, Ill.).

Burger, E. On extrema with side conditions. Econometrica **23** (1955), 451–452.

A proof is given of sufficient conditions for a relative minimum of a function $f(x_1, \cdots, x_n)$ where the point x is subject to constraints $g_k(x_1, \cdots, x_n)=0$, based on a determinantal condition for the positive definiteness of a quadratic form whose variables are subject to linear constraints. It seems to the reviewer that a direct continuity proof is simpler, as given, e.g., by Bliss [Lectures on the calculus of variations, Univ. of Chicago Press, 1946, pp. 211, 212; MR **8**, 212] with no restriction on the rank of the matrix $(\partial g_k/\partial x_i)$. *L. M. Graves* (Chicago, Ill.).

Marx, Imanuel. On the classification of unstable minimal surfaces with polygonal boundaries. Comm. Pure Appl. Math. **8** (1955), 235–244.

Let γ designate a simple closed polygon of N vertices in Euclidean 3-space, and let $P_1, P_2, \cdots, P_N, P_{N+1}=P_1$ represent its vertices in cyclic order. Let $p_1, \cdots, p_N, p_{N+1}=p_1$ denote points on the real axis of the (u, v) plane, arranged cyclically. According to M. Shiffman, a surface defined by a vector function $\mathbf{x}(u, v)$ in the upper half-plane is said to be a quasi-minimal surface bounded by γ if (i) $\mathbf{x}(u, v)$ maps each segment $\overline{p_i p_{i-1}}$ into the infinite segment defined by $\overline{P_i P_{i-1}}$, and (ii) $\mathbf{x}(u, v)$ minimizes the Dirichlet integral

$$D[\mathbf{x}]=\iint_{v>0}\{|\mathbf{x}_u|^2+|\mathbf{x}_v|^2\}dudv$$

among all functions satisfying (i). Shiffman has proved that for a given parameter set $\{p\}=\{p_1, \cdots, p_N\}$ there is a uniquely defined quasi-minimal surface determined by γ and that for this surface the Dirichlet integral $D[\mathbf{x}]=d(p)$ is infinitely differentiable with respect to the parameters.

Let $\{p^*\}$ designate the parameter set corresponding to a minimal surface which spans γ and is represented isothermally on the upper half (u, v)-plane. (Three parameter values are assumed fixed.) The index of the quadratic form

$$q(s)=\sum_{i,j=1}^{N-3}\frac{\partial^2 d(p^*)}{\partial p_i \partial p_j}s_i s_j$$

may be used to classify unstable minimal surfaces bounded by γ, considered as critical quasi-minimal surfaces.

Another possibility is to consider the second variation of the surface area of a given minimal surface bounded by γ, with respect to variation along the surface normal. For a minimal surface $\mathbf{z}(u, v)$ represented isothermally on $v>0$, the second variation of surface is

$$E[f]=\iint_{v>0}(f_u^2+f_v^2+2kwf^2)dudv,$$

where $f(u, v)$ is the normal variation, k is the Gaussian curvature, and $w=|\mathbf{z}_u|^2=|\mathbf{z}_v|^2$. This leads to another classification according to the number of negative characteristic values of the Euler expression

$$L[f]=-\nabla^2 f+2kwf$$

associated with $E[f]$.

The main result of this paper is that under suitable assumptions these two numbers are equal. The proof depends on the construction, by means of suitable variations, of a new functional $Q(s)$ which is equal to $q(s)$ and differs from $E[f]$ by a non-negative quantity. The result then follows from a proper choice of the functions which define the variation.

The reviewer was unable to follow all details of the proof, in particular, the statement that $Q(s)=q(s)$. *R. Finn* (Los Angeles, Calif.).

Theory of Probability

Castañs Camargo, Manuel. A theory of certitude. I. Definitions and properties. An. Real Soc. Españ. Fis. Quim. Ser. A. **51** (1955), 215–232. (Spanish. English summary)

"A quantity is derived which will measure the certainty of the occurrence of a phenomenon or of the measure of a magnitude. The properties of such a quantity are studied for both discrete and continuous distributions." [Author's summary.]

The quantity adduced is

$$\log n+\sum_1^n p_i \log p_i$$

for a distribution on n events and

$$\log (a-b)+\int_a^b P(x)\log P(x)dx$$

for a distribution on $[a, b]$. Some importance is attached to the small differences between these expressions and the usual ones for entropy and information. *L. J. Savage.*

Popper, Karl R. Two autonomous axiom systems for the calculus of probabilities. British J. Philos. Sci. **6** (1955), 51–57.

In some theories of probability $P(E|F)=1$ whenever (i) F implies E, and (ii) $P(F)\neq 0$, where E and F are propositions. The author suggests two systems of axioms for which the condition $P(F)\neq 0$ is not required, and he states that each of these systems is demonstrably consistent. The main motivations are applications (i) to formal logic, and (ii) to circumstances in which F is a universal law of nature of zero probability. (The author claims, in private correspondence, that he has maintained since 1934 that universal laws have zero probability, and that he can prove it. Until the proof is published the

matter will be controversial. Another controversial matter is the implicit assertion that the addition axiom is not conventional.) *I. J. Good* (Cheltenham).

Blackman, Jerome. On the approximation of a distribution function by an empiric distribution. Ann. Math. Statist. 26 (1955), 256–267.

Consider independent random variables $x_1, \cdots, x_n$ with common distribution function $F(s)$ and empiric distribution function $F^*(x)$. Let a_n be the value of a which minimizes

$$H(a)=\int_{-\infty}^{+\infty}|F(x-a)-F^*(x)|^2dF(x-a).$$

If the first three derivatives of $F(x)$ are continuous and bounded, then $n^{\frac{1}{2}}a_n$ is asymptotically normally distributed with mean 0 and a variance depending on the original $F(x)$. *S. W. Nash* (Vancouver, B.C.).

Wartmann, Rolf. Einige Bemerkungen zur logarithmischen Normalverteilung. Mitteilungsbl. Math. Statist. 7 (1955), 152–165.

If $x=\beta e^{z/n}+\alpha$, where z is distributed according to $N(\mu', \sigma'^2)$, x is said to be log-normally distributed. The author notes that if the mean μ and variance σ^2 of x are held fixed as $\alpha\to-\infty$ the distribution of x tends to the Gaussian distribution. He uses this to derive an approximation that expresses x linearly in terms of z and of γ_1, the third moment of the distribution of x. Expressions are given for the higher moments of x and also large sample formulae for the variance of estimates of the unknown parameters $(\mu, \sigma^2, \gamma_1)$ obtained from sample cumulants.

D. G. Chapman (Seattle, Wash.).

Fagen, R. E., and Riordan, John. Queueing systems for single and multiple operation. J. Soc. Indust. Appl. Math. 3 (1955), 73–79.

The authors consider a queueing situation with a Poisson input, n servers, a general service-time distribution $dF(s)$ and two alternative queue-disciplines: (1) "single operation": one server is allotted to each customer, customers being dealt with in the order of arrival; (2) "multiple operation": each customer is attended to simultaneously by all n servers, customers being dealt with in the order of arrival and a customer's service being terminated as soon as some server has finished with him. Thus case (1) is the standard queueing problem with a Poisson input, n servers and a service-time distribution $dF(s)$, while case (2) can be identified with a standard queueing problem with a Poisson input, one server and a service-time distribution $dF_n(s)$, where $1-F_n(s)=[1-F(s)]^n$. It should be noted that case (2) is a genuine alternative to case (1) only when the length of each service time is a random variable associated with the server rather than with the customer, so that the multiplicity of servers will reduce s_1 to min $(s_1, s_2, \cdots, s_n)$. The authors compare (1) and (2) by calculating the expected values of the "elapsed time" (=waiting time+service time). [To minimise this in a stationary system is equivalent to minimising the average queue-size.] The authors report that (1) is better (in this sense) than (2) if s is fixed and not a random variable, while (2) is preferable if s has a negative-exponential distribution. For certain intermediate forms of service-time distribution (1) is the better for heavy and (2) for light loading. *D. G. Kendall.*

Brigham, Georges. On a congestion problem in an aircraft factory. J. Operations Res. Soc. Amer. 3 (1955), 412–428.

This is a study of the optimum number of clerks in a stockroom, balancing the cost of clerks against the cost of customers' waiting time. The queuing model, set up after making observations on the relevant random variables, is one of random arrivals, exponential service time, and random service. The behaviour of stockroom clerks, as noticed by the author, is similar to that of long-distance telephone operators as reported by R. I. Wilkinson [Bell System Tech. J. 32 (1953), 360–383]; for few arrivals, service in order of arrival is possible, but as the number waiting increases, random service becomes inevitable. As a further refinement, the distribution of idle time of clerks, again on the assumption of random assignment, is formulated; if this time is usable otherwise, it is not properly chargeable to stockroom service. Here two conditional probabilities may be formulated: (i) the probability that a given server idle at a given epoch of state x (x servers busy) will continue idle for at least t and (ii) the probability that a server who has just become idle at a given epoch of state x will continue idle for at least t. The author seems to use the second since he finds the differential recurrence relation: $g_x'(t)=-ng_x(t)/(c-x)$, with n the average number of arrivals, c the number of servers, $g_x(t)$ the probability in question, $g_x'(t)$ its time derivative, whereas the first (with h the average service time) leads to

$$hf_x'(t)=nh(c-x-1)f_{x+1}(t)/(c-x)+xf_{x-1}(t)-(a+x)f_x(t).$$

But he also uses the state probabilities going with the first: $P_x=(a^x/x!)P_0$, $a=nh$; for the second the probability of state x is $aP_x/(x+1+a)$. It may be noticed that the second conditional probability is related to the first much as the Mellor approximation to random service conditional delay probability is related to the exact result.

J. Riordan (New York, N.Y.).

Newell, G. F. Statistical analysis of the flow of highway traffic through a signalized intersection. Quart. Appl. Math. 13 (1956), 353–369.

An analysis is made of the delays produced by a traffic light on a one-lane unidirectional traffic. The postulates include assumptions to the effect that cars arrive at the intersection at least a certain minimum time δ apart, and leave at least the same time δ apart. A set of integral equations is obtained for the "equilibrium" distribution $P(t|y)$, the probability that a car leaves at time y, if it arrived at time t, on the assumption of independent interarrival times. Approximate solutions for the delay distributions are obtained, and graphed for the cases (i) where the interarrival times are equal, and (ii) where the interarrival times have a density $\alpha e^{-\alpha(t-\delta)}$ for $y>\delta$, 0 for $y<\delta$. As might be expected, even the approximate formulas are quite complicated. *E. Reich* (Santa Monica, Calif.).

Lee, C. Y. Analysis of switching networks. Bell System Tech. J. 34 (1955), 1287–1315.

A switching network, mathematically speaking, is a linear graph whose line elements are switches, which have two states, open and closed, hence are elements of a Boolean algebra when their behavior in time is ignored; the graph has two (outer) terminals, the input and output. Its time behavior, the sequence of opening and closing of switches, is strictly determined by the control system associated with it. In networks with a large number of

switches, which are common in telephony, a strict analysis is impossible, and the idea followed in the present paper is to represent the behavior by assuming the switches to be stationary time-dependent random variables. The network is then a random variable of the same kind which is a function of its elements. While elements strictly speaking are correlated, the formal work for the most part assumes independence. With independence, it is possible to write a generating function for the number of connected paths and hence for the expectation of a connection, a result useful in blocking studies, which are made for fairly elaborate multistage networks. Finally, some results are given for "retrial" distributions, which are conditional probabilities that a network with a connecting path at some given time is in the same condition at a later time. *J. Riordan.* (New York, N.Y.).

See also: Urbanik, p. 444.

Mathematical Statistics

Bennett, B. M. The cumulants of a sample mean from a finite population of first N integers. Trabajos Estadist. **6** (1955), 31–32. (Spanish summary)

Irwin, J. O. A unified derivation of some well-known frequency distributions of interest in biometry and statistics. J. Roy. Statist. Soc. Ser. A. **118** (1955), 389–398; discussion, 399–404.

Murty, V. N. The distribution of the quotient of maximum values in samples from a rectangular distribution. J. Amer. Statist. Assoc. **50** (1955), 1136–1141.

Rider, Paul R. The distribution of the product of maximum values in samples from a rectangular distribution. J. Amer. Statist. Assoc. **50** (1955), 1142–1143.

Simon, Herbert A. On a class of skew distribution functions. Biometrika **42** (1955), 425–440.

Distributions of word-frequencies, city sizes, incomes and biological genera (by number of species) are all J-shaped or highly skewed. They may be approximated, at least in the long upper tails, by the function

$$f(i)=(a/i^k)b^i \quad (i=1, 2, \cdots), \tag{1}$$

where a, b, k are constants with b very close to unity and $k>1$. The author suggests a stochastic model of some generality leading to (1), and discusses applications to the above empirical distributions. *H. A. David* (Melbourne).

Huzurbazar, V. S. Exact forms of some invariants for distributions admitting sufficient statistics. Biometrika **42** (1955), 533–537.

Jeffreys [Proc. Roy. Soc. London. Ser. A. **186** (1946), 453–461; MR **8**, 163] suggested as (invariant) measures of "distance" between two (Lebesgue) density functions p and q the integrals of $(p-q)\log(p/q)$ and $|p^{1/m}-q^{1/m}|^m$ (m is positive and even, below). The author computes these in the case where p and q are members of a (multiparameter) Koopman-Darmois family. *J. Kiefer.*

Mauldon, J. G. Pivotal quantities for Wishart's and related distributions, and a paradox in fiducial theory. J. Roy. Statist. Soc. Ser. B. **17** (1955), 79–85.

The main mathematical result is the following theorem. Given a random sample of size $n+1$ drawn from a normal p-variate population, let A and α be the dispersion matrices of the sample and the population respectively. Suppose $n\geqq p$ and let L and λ be the unique non-singular positive upper-triangular matrices satisfying $L'L=A$ and $\lambda'\lambda=\alpha$. Finally let $K=L\lambda^{-1}$. Then the elements K_{ui} of K ($u\leqq i$) are independently distributed, as standard normal deviates if $u<i$ and as $\sqrt{(\chi_{n+1-i}^2)}$ if $u=i$. Wishart's distribution is deduced. The exciting point in the paper is the statistical consequence of the theorem. K is a function of the sufficient statistics and the parameters α, whose distribution does not involve the parameters. It is therefore a pivotal quantity for the application of Fisher's theory of fiducial probability [Ann. Eugenics **6** (1935), 391–398, in particular page 395, and many later references] and the fiducial distribution of α can be found. The author remarks that if we apply a non-singular linear transformation to the sample (for example, merely reordering the variates) and reapply the theorem, a different pivotal quantity and a different fiducial distribution will be obtained. A simple example is given for $n=p=2$ and illustrated numerically. This result shows that the fiducial method is not a "process of reasoning by which we may pass, without arbitrariness or ambiguity, from forms of statements in which observations are regarded as random variables, having distribution functions involving certain fixed but unknown parameters, to forms of statement in which the observations constitute fixed data, and frequency distributions are found for the unknown parameters regarded as random variables" [Fisher, Sankhyā **7** (1945), 129–132, page 130; MR **7**, 318]. Further restrictions will have to be placed on the pivotal quantities before this highly attractive technique can have the uniqueness property. *D. V. Lindley.*

Babbar, M. M. Distributions of solutions of a set of linear equations (with an application to linear programming). J. Amer. Statist. Assoc. **50** (1955), 854–869.

The author considers the solution of a set of linear equations with errors in the coefficients. The equations are $(B+b)X=(Q+\varepsilon)$, where X is the vector to be determined and b and ε represent random errors, here assumed normal and uncorrelated. Procedures are presented to determine approximate probability limits on each X_i and on Y, a linear function of the X_i; also the approximate probability that X_i and Y would exceed or be less than given constants.

The results are applied to an example of linear programming based on 25 years of crop data from a group of Iowa farms. The B's are input coefficients; the Q's are amounts of various inputs used on each crop; and Y is gross profit. In the example, the ε's are assumed to be zero.

The X's are ratios of random variables, i.e. $X=N/D$. If one sets $X^*=E(N)/E(D)$, then $u=DX^*-N$ is normally distributed with zero mean and variance

$$V(u)=V(D)X^{*2}-2CV(N, D)X^*+V(N),$$

where V refers to variance and CV to covariance. Confidence limits can be obtained by solving the quadratic equation $u^2/V(u)=t^2$ for X^* in terms of estimates of $V(D)$, $V(N)$ and $CV(N, D)$ and the value of Student's t for the selected confidence coefficient. The author prefers to state the problem in terms of limits on X with means and variances estimated from the data and t being considered as a normal deviate. This reviewer wishes the author had supplied estimates of the correlations of the

input coefficients to substantiate his assumption that such correlations were essentially zero. *R. L. Anderson.*

Cohen, A. Clifford, Jr. Maximum likelihood estimation of the dispersion parameter of a chi-distributed radial error from truncated and censored samples with applications to target analysis. J. Amer. Statist. Assoc. 50 (1955), 1122–1135.

Bailey, Norman T. J. Some problems in the statistical analysis of epidemic data. J. Roy. Statist. Soc. Ser. B. 17 (1955), 35–58; discussion 58–68.

In this paper the author makes an extensive exploration of the simple models of intra-familial epidemics, reviewing and extending earlier work, particularly his own. Two essentially distinct models are considered. The first, the continuous infection model, assumes that initially the epidemic begins when a infectives are introduced into a group of n susceptibles. If at a later time there are r susceptibles still uninfected and s infectives in circulation, the chance of one new infection in the time interval $(t, t+dt)$ is $\beta rs\,dt$ and the chance of one removal in the same interval $\gamma s\,dt$. The distribution of the total number of cases and the use of this random variable to estimate γ/β was previously given [Biometrika **40** (1953), 177–185; MR **14**, 1101]. This is now extended: it is shown that if in families of size two removal times are also observed, maximum likelihood estimates of both β and γ are easily derived.

The second model assumes that the infectious period is reduced to a point while the latent period of development of the disease is constant. This gives rise to what has been called the chain-binomial model studied by many authors. Maximum-likelihood estimators of the probability of infection are given for this model when the actual "chains" of infection are observed and also when only the total size of the epidemic is known. The analysis is also extended to cases where there is a variable chance of infection between families, as well as between individuals. In these problems a special parametric form is assumed for the distributions of the variable chance of infection and maximum likelihood estimates are derived for short chains, i.e. from data on epidemics in families of two or three. All of the probabilities involved, can in these cases be written down explicitly from elementary considerations. Numerical examples are given, exhibiting the models fitted to actual data. Many though not all of the fits are satisfactory. The subsequent discussion is valuable: in particular, attention is drawn to the remarks of D. G. Kendall. He suggests a combination of a birth and death process and a deterministic model as giving a useful approximation to the continuous infection stochastic model and he also makes some points about the threshold theorem of Kermack and McKendrick. *D. G. Chapman.*

★Hemelrijk, J. Exemple d'application des méthodes non paramétriques et un nouveau test pour l'égalité de plusieurs probabilités. Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 93–111. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Suppose $\Omega=\{\omega\}$ is the set of all possible $(m+n)$-tuples, ω, of m zeros and n ones and $X=(X_1, \cdots, X_{m+n})$ is a random vector taking on its values in Ω. Let H be the hypothesis that $P(X=\omega)=\binom{m+n}{n}^{-1}$ for all ω in Ω. Let $Q_1, \cdots, Q_{m+n}$ be constants and $T=\sum_{i=1}^{m+n} X_iQ_i$. The paper discusses several new situations in which it is reasonable to use a statistic of the form T to test a hypothesis which implies H. The distribution of T is discussed. The main examples refer to a test for the competence of an inspector and to a test for the equality of the success probabilities in n independent success-failure experiments. *M. Dwass* (Stanford, Calif.).

★van Dantzig, D. Sur les ensembles de confiance généraux et les méthodes dites non paramétriques. Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 73–91. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Suppose Ω is a family of distribution functions, H the hypothesis that the distribution of a random vector $X=X_1, \cdots, X_n$ is contained in $\Omega_0 \subset \Omega$, and $t(X)$ a statistic. H is rejected if t is large. A method is given for "approximating" a confidence set for $(E_\theta t, \mathrm{Var}_\theta t)$, θ in Ω. A discussion is given of test statistics t, useful in testing nonparametric hypothesis, for which this method might be applicable. *M. Dwass* (Stanford, Calif.).

Williams, E. J. Significance tests for discriminant functions and linear functional relationships. Biometrika **42** (1955), 360–381.

The author continues the studies of himself [Biometrika **39** (1952), 17–31; MR **14**, 299] and Bartlett [Ann. Eugenics **16** (1951), 199–214; MR **13**, 666] on exact tests, based on the factorization of likelihood ratios, for the adequacy of a hypothetical linear discriminant; and shows how they may be modified to yield exact tests for an assumed linear functional relationship amongst the variables. The connection between the two problems lies in the fact that the assumption of the adequacy of a single discriminant for a p-variable population is equivalent to the assumption of $p-1$ linear relations among the means of the p variates; conversely, the assumption of a single linear relationship is equivalent to the assumption of the adequacy of $p-1$ discriminants. The paper is divided into sections dealing with the following topics. I. Formulation of the problem in terms of p "investigational" variates X_i with a multivariate normal distribution, and q "instrumental" variates Y_i (Reiersøl's terminology) upon which the X_i have a linear regression, and a statement of preliminary results. II. The factorization of the likelihood ratios in terms of determinants. III. The expression for the tests in terms of canonical correlations, and then in terms of a covariance analysis of the X_i, eliminating the hypothetical discriminant $\xi=\sum l_iX_i$. (This latter approach has the merit that it is not necessary to calculate latent roots.) IV. The corresponding analysis for a linear functional relationship, accompanied by some general remarks on the nature of linear functional relationships. The generalizations for $r(>1)$ discriminants (or functional relationships) is indicated. *D. V. Lindley* (Cambridge, England).

Good, I. J. The likelihood ratio test for Markoff chains. Biometrika **42** (1955), 531–533.

The author discusses the limiting distribution of the likelihood-ratio test and several variants of it, for testing whether a Markov chain is of order μ or ν $(\mu<\nu)$, following the considerations of Hoel in the case $\nu=\mu+1$ [Biometrika **41** (1954), 430–433; MR **16**, 498]. *J. Kiefer.*

Hannan, E. J. An exact test for correlation between time series. Biometrika **42** (1955), 316–326.

The author extends the methods of an earlier paper [Biometrika **42** (1955), 133–142; MR **16**, 1040] and ob-

tains exact tests of the correlation between two series which are generated by various simple regressive and autoregressive models. *P. Whittle* (Wellington).

Watson, G. S. Serial correlation in regression analysis. I. Biometrika **42** (1955), 327–341.

The author considers the consequences of assuming an incorrect covariance matrix for the residuals from regression of a set of observations. He deduces inequalities on the bias in the estimates of variance of the regression coefficients, on the efficiency of the estimates of the regression coefficients, and on the significance points of the various t- and F-tests. *P. Whittle.*

Foster, F. G., and Teichroew, D. A sampling experiment on the powers of the records tests for trend in a time series. J. Roy. Statist. Soc. Ser. B. **17** (1955), 115–121.

A technical account of how an electronic computer was used in the sampling experiment reported in an earlier paper by Foster and A. Stuart [same J. **16** (1954), 1–22; MR **16**, 385]. In the experiment $4 \cdot 10^6$ random normal deviates were generated by the machine. *H. Wold.*

Theory of Games, Mathematical Economics

Luce, R. Duncan. k-stability of symmetric and of quota games. Ann. of Math. (2) **62** (1955), 517–527.

The notion of k-stability for n-person games, where $0 \leq k \leq n-2$, introduced previously by the author [Ann. of Math. (2) **59** (1954), 357–366; MR **15**, 975] is applied to symmetric and quota games. Let $m(S)$ denote the characteristic function of a game with players $1, \cdots, n$ normalized so that $m(\{i\})=0$ and $m(I_n)=1$, where $I_n=\{1, \cdots, n\}$. Theorem: A symmetric game is k-stable if and only if $M(S) \leq i/n$ for $0 \leq i \leq k+1$ and every i-person coalition S. A game is quota [Shapley, Contributions to the theory of games, v. 2, Princeton, 1953, pp. 343–359; MR **14**, 779] if there exists a real n-tuple $Q=\|q_i\|$ such that $\sum_{i=1}^{n} q_i=1$ and $m(\{i, j\})=q_i+q_j$ for $i \neq j$. Player i is called weak if $q_i<0$. Theorem: A quota game is 1-stable if and only if there is no weak player. Several more detailed results are obtained concerning k-stability of quota games, particularly for the case $k \geq 2$. *W. H. Fleming.*

Suppes, Patrick, and Winet, Muriel. An axiomatization of utility based on the notion of utility differences. Management Sci. **1** (1955), 259–270.

Let K be a set, to be thought of as alternative events, with a binary relation Q (for xQy read: x is at least as desirable as y) and a quaternary relation R (for $(x, y)R(z,w)$ read: the difference in utility between x and y is at least as great as the difference in utility between z and w). The authors list a set of eleven axioms on the relations Q and R and show that if these axioms are satisfied then there exists a (utility) function φ, unique up to a linear transformation such that (i) $\varphi(x) \geq \varphi(y)$ if and only if xQy, (ii) $|\varphi(x)-\varphi(y)| \geq |\varphi(z)-\varphi(w)|$ if and only if $(x, y)R(z, w)$. The proof of this result is due to H. Rubin. *D. Gale* (Providence, R.I.).

★ **Georgescu-Roegen, Nicholas. Limitationality, limitativeness, and economic equilibrium.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 295–330. National Bureau of Standards, Washington, D.C., 1955.

A factor of production is defined as limitative if its increase is both necessary and sufficient for an increase in output. When one factor is limitative, obviously all others must have zero marginal productivity. Suppose the factor endowments of a country are such that land is limitative and labor is redundant. Competitive equilibrium at full employment would require the real wage to fall to zero. If there is a positive conventional minimum wage or subsistence level, profit maximization will lead to unemployment and less than maximal output. Output could be increased by abandoning the marginal productivity or profit maximization principle in favor of some alternative allocation rules. The author develops this proposition in a Leontief model and in a more general set-up, and uses it to discuss some economic and social patterns observable in overpopulated countries and elsewhere. *R. Solow.*

★ **Beckmann, M., and Marschak, T. An activity analysis approach to location theory.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 331–379. National Bureau of Standards, Washington, D. C., 1955.

Straightforward application of linear activity analysis to: I) the problem of a firm operating branch plants at several locations, using materials obtained from various deposits, and selling product at various markets; and II) the characterization of the efficient commodity-flow programs for an economy composed of regions. The interpretation of the dual variables as internal prices is stressed. *R. Solow* (Cambridge, Mass.).

See also: Foulkes, p. 432.

Mathematical Biology

Malécot, G. Sur les modèles stochastiques linéaires asymptotiquement stationnaires. Ann. Univ. Lyon. Sect. A. (3) **17** (1954), 19–35.

Let $B(n, p)$ denote the binomial distribution with the indicated parameters. Then, if r_0 is a fixed integer such that $0 \leq r_0 \leq n$, the scheme $\text{distr}(r_{j+1}|r_0, r_1, \cdots, r_j)=B(n, r_j/n)$ for $j=0, 1, 2, \cdots$ can be used to describe the fluctuations in the number r_j of copies of a particular gene A in the jth generation, when the population is fixed so as to contain in each generation n homologous gene-loci. One then has

$$\mathcal{E}(r_{j+1}|r_j)=r_j \text{ and } \text{var}(r_{j+1}|r_j)=r_j(1-r_j/n)$$

and the probabilities of ultimate fixation at $r_j=0$ or $r_j=n$ can now be evaluated in the usual way [R. A. Fisher, The genetical theory of natural selection, Oxford, 1930; cf. also S. Wright, Statistical genetics in relation to evolution, Hermann, Paris, 1941; and W. Feller, Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 227–246; MR **13**, 671]. Similarly when mutation rates $\lambda(a \to A)$ and $\mu(A \to a)$ are in operation one can write

$$\text{distr}(r_{j+1}|r_0, r_1, \cdots, r_j)=B(n, (1-\mu)r_j/n+\lambda(1-r_j/n)),$$

and study the steady-state gene-frequency distribution attained as $j \to \infty$. The author's method leads to the characteristic values previously found for this problem by Feller (paper cited above) and to an exact expression for the steady-state variance of the gene-frequency (generalising a result of Wright). More generally the author considers the analogous problem involving R sub-popu-

lations linked by migration. One now has

$$\mathcal{E}(r_{j+1}^{s}|r_j^1, r_j^2, \cdots, r_j^R)=\sum_{i=1}^{R} l_i^s r_j^i$$

and

$$\operatorname{cov}(r_{j+1}^{s}, r_{j+1}^{\sigma}|r_j^1, r_j^2, \cdots, r_j^R)=\phi^{s\sigma}(r_j^1, r_j^2, \cdots, r_j^R),$$

where s, σ and t identify the sub-populations and the l's are given constants and the ϕ's given functions. The author develops a general technique for handling problems of this kind and deduces solutions to further problems of importance in genetics. *D. G. Kendall* (Oxford).

Haimovici, A. Le problème de l'évolution de deux espèces en conditions spéciales. Acad. R. P. Române. Fil. Iaşi. Stud. Cerc. Şti. 1 (1950), 129–144. (Romanian)

L'A. étudie par la méthode de Volterra l'interdépendance entre les nombres d'individus x et y de deux espèces et d'un produit z sécrété par les individus de l'une des deux espèces. Les équations décrivant l'interdépendence sont écrites sous la forme

$$\frac{dx}{dt}=x\Big[\varepsilon_1-\gamma_1 y-\delta_1 z-\int_0^\infty F_1(\tau)y(t-\tau)d\tau\Big],$$

$$\frac{dy}{dt}=y\Big[-\varepsilon_2+\gamma_2 y+\delta_2 z+\int_0^\infty F_2(\tau)x(t-\tau)d\tau\Big],$$

$$\frac{dz}{dt}=\alpha_1 x-\alpha_2 y+\int_0^\infty [F_3(\tau)x(t-\tau)-F_4(\tau)y(t-\tau)]d\tau.$$

On démontre que ce système a une intégrale unique, si l'on connait les fluctuations des espèces et de la quantité de produit, pendant la période de l'hérédité (§ 1, 2 et 3). On étudie ensuite les petites fluctuations autour de l'état stationnaire, s'il y a un tel état (§ 4). Dans le cas particulier où les quantités x, y et z tendent vers des limites pour $t=\infty$, alors ou bien ces limites sont les valeurs stationnaires, ou bien les espèces se détruisent, tandis que la quantité z tend vers le plus grand des rapports ε_1/δ_1, ε_2/δ_2 (§ 5 et 6). Dans le même cas, où les quantités x, y, z tendent vers les limites, les valeurs moyennes des espèces, pendant une durée qui tend vers l'infini, sont précisément les valeurs stationnaires (§ 7). Si l'on détruit un nombre d'individus des deux espèces, proportionnellement au nombre total des individus de ces espèces et uniformément dans le temps, le rapport des moyennes des deux espèces reste constant, tandis que la moyenne de z décroit. Enfin si l'on extrait une quantité de produit z, les moyennes des espèces restent inchangées, tandis que la moyenne de z croit avec cette même quantité (§ 8).

Autoreferat (Zbl 41, 432).

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Have

★**Montgomery, Deane, and Zippin, Leo. Topological transformation groups.** Interscience Publishers, New York-London, 1955. xi+282 pp. $5.50.

Almost two decades have passed since publication of the first edition of Pontrjagin's "Topological groups" [Moscow, 1938; Princeton, 1939; MR 1, 44] which has been since considered as one of the standard reference books in the field. In the meantime, the theory of topological groups has made outstanding progress, culminating in the solution of Hilbert's fifth problem by Gleason and by the authors of the present book. The authors give here a detailed account of those important results on locally compact topological groups obtained in this period, suggesting at the same time further future developments in the theory. The book is divided roughly into three parts; in the first two chapters, the authors give those results on topological groups which were obtained in what they call the classical period (ending around 1935), thus preparing for the succeeding chapters. The next two chapters are devoted to the study of the structure of locally compact groups which leads to a solution of Hilbert's problem. In the last two chapters, the authors discuss the properties of transformation groups acting on various spaces. Here the results are not yet as conclusive as those of the preceding chapters, but even so, the reader may find there fundamental theorems on transformation groups which the reader himself may use as a basis for further investigation of this yet unexploited territory in mathematics. More details of the contents follow.

In Chap. I the authors start with the definition of topological spaces and introduce such fundamental notions as topological groups, homogeneous spaces, transformation groups, etc.. Tychonoff's theorem and the theorem on the existence of invariant metric for certain topological groups are proved. In Chap. II various properties of locally compact groups are discussed and, in particular, the following theorems are proved: 1) every locally compact group can be approximated, in a certain sense, by locally compact separable metric groups; 2) if G is a locally compact transformation group acting on a locally compact separable space and if the factor group of G modulo its connected component of e satisfies a certain condition (e.g., if the factor group is trivial), then G is also separable; 3) v. Neumann's theorem that a locally compact group is a Lie group if it can be mapped into a linear group by a continuous one-one homomorphism; 4) the existence of Haar measures on locally compact groups. Many interesting examples of transformation groups are also given.

In Chaps. III, IV, the authors study the structure of locally compact groups following the method of Gleason-Yamabe. First, the existence of nontrivial one-parameter subgroups in a (metric, connected, non-trivial) locally compact group G is proved. Assuming then that G has no small subgroups (i.e., that G has a neighborhood of e containing no non-trivial subgroups of G), properties of the family of those (local) one-parameter subgroups of G are carefully studied and there is made a finite-dimensional real vector space M on which G acts as group of linear transformations. The kernel N of the so-obtained representation of G on M is then proved to be an abelian Lie group. On the other hand, Kuranishi's extension theorem is also proved, which says that if a locally compact group G has a closed normal subgroup N such that N and G/N are both Lie groups, then G itself is a Lie group. Applying this to the above G and N, it then follows that a locally compact connected group without small subgroups is a Lie group. Returning to general locally compact groups, another key lemma is proved, namely that if U is any neighborhood of e in a locally compact group G, there exists a subgroup H contained in U and a neighborhood W of e in G such that any subgroup of G contained in W is a subgroup of H. All these results are then combined to prove the following fundamental theorem on locally

compact groups: let G be a locally compact group such that G/G_0 is compact, G_0 being the connected component of e in G, and let U be an arbitrary neighborhood of e in G. Then there is a compact normal subgroup H of G such that G/H is a Lie group. Analysing the topological structure of such groups, it then follows from the above theorem that a locally euclidean topological group is a Lie group, a solution of Hilbert's fifth problem.

As mentioned above, in Chaps. V, VI, the authors give various interesting results on transformation groups, most of which were obtained by the authors themselves. Some of those results will be given below. 1) Let G be a locally compact effective transformation group of a connected manifold of class C^1 and let each transformation of G be of class C^1. Then G is a Lie group. 2) Let G be a compact Lie transformation group acting on a Hausdorff space M and let p be a point in M with a neighborhood X such that, if $x \in X$ and $a, b \in G$, then $a(x)=b(x)$ only when $a=b$. Then the orbits of G on M have a local cross-section at p. 3) If G is a connected Lie group which acts transitively on a compact simply connected manifold M, then G contains a compact subgroup which also acts transitively on M. 4) Let G be a locally compact group acting effectively on the line or the circle. Then G is a Lie group. 5) Let G be a connected locally compact group and H a closed subgroup of G with finite-dimensional G/H. Then H contains a closed connected normal subgroup K of G such that G/K is also finite dimensional. 6) If a compact group G acts effectively on a connected locally euclidean space E and if all the orbits in E are locally connected, then G is a Lie group. 7) Let G be a compact connected Lie group which acts effectively on a connected locally euclidean space E and let k be the highest dimension of any orbit. Then $\dim G \leqq k(k+1)/2$. 8) Let G be a compact zero-dimensional group which acts effectively on a two-dimensional connected locally euclidean space E. Then G is finite. 9) Let G be a compact group (non-trivial) which acts effectively on euclidean three-space E. Then G is a circle group or the proper orthogonal group in three variables and the action of G is topologically equivalent respectively to the group of all rotations around an axis or the group of all proper orthogonal transformations. *K. Iwasawa*

Gĕtc [Götz]. A. On separability of topological groups. Fund. Math. **42** (1955), 55–56; addendum, 356. (Russian).

The author proves that a locally compact group is separable if and only if its Haar measure is separable by showing first that a locally compact group with separable Haar measure is metrizable, and next that a locally compact metric space with separable regular Borel measure such that open sets have positive measure is separable. The addendum calls attention to the fact that the main result is known [Halmos, Measure theory, McGraw-Hill, New York, 1950, p. 268, Ex. 4; MR **11** 504].

G. K. Kalisch (Princeton, N.J.).

Schieferdecker, Eberhard. Zur Einbettung metrischer Halbgruppen in ihre Quotientenhalbgruppen. Math. Z. **62** (1955), 443–468.

The algebraic apparatus of this paper is essentially that of Murata [Osaka Math. J. **2** (1950), 1–5; MR **12**, 155] whose concepts and notations we shall use in this review (the object of Murata's definition 2 will be called $Q_l(H;m)$). The first two sections contain an exposition of the algebraic preliminaries including results of Murata, a generalization of Theorem 1 of Vandiver [Amer. J. Math. **62** (1940), 72–78; MR **1**, 105] and results about "Ore semigroups" [Ann. of Math. (2) **32** (1931), 463–477; an Ore semigroup is a regular semigroup such that any two elements have a common left multiple]. Section 3 contains a necessary and sufficient condition for the extendibility of an invariant metric on a semigroup H without identity to the semigroup H^* obtained by adjoining an identity to H. The next two sections are devoted to the topic mentioned in the title. Since semigroups with invariant metrics are regular, the concept of m-left-invariant is introduced (for regular $m \subset H$): a metric $|a, b|$ on H is so called if for all a and b in H and α and β in m $|\alpha a, \beta b| = |a, b|$. First, necessary and sufficient conditions for the extendibility of an m-left-invariant metric from H to $Q_l(H; m)$ (and allied metrics) are given. These are then applied to proving: that an invariant metric of a commutative regular semigroup can be extended to its quotient group [generalizing Theorem 15 of Gelbaum, Kalisch, Olmsted, Proc. Amer. Math. Soc. **2** (1951), 807–821; MR **13**, 206]; an analogous statement concerning Ore semigroups; two additional theorems dealing with invariant metrics satisfying various additional conditions. The last section is devoted to a comparison of the author's process of embedding semigroups in quotient semigroups with that found in the Gelbaum, Kalisch, Olmsted article referred to above — showing the lack of uniqueness of the solution of the embedding problem of topological semigroups in their quotient semigroups. *G. K. Kalisch.*

See also: Schieferdecker, p. 346; Hartman, p. 362; Edwards, p. 387; Brownell, p. 388.

Lie Groups, Lie Algebras

★ **Séminaire "Sophus Lie" de l'Ecole Normale Supérieure, 1954/1955. Théorie des algèbres de Lie. Topologie des groupes de Lie.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. 200 pp. (polycopiées). 1200 francs.

The contributors to this seminar are M. Berger, A. Blanchard, F. Bruhat, P. Cartier, M. Lazard, and J-P. Serre, Cartier carrying by far the heaviest load (14 out of the total of 24 lectures). The aims of the seminar and the spirit in which they were pursued are described the preface, as follows: "Notre Séminaire s'était donné pour but de donner un exposé aussi complet que possible du sujet énoncé. Nous avons voulu, par ailleurs, donner des démonstrations complètes et aussi élémentaires que possible, n'assumant comme connaissances que le bagage usuel d'un mathématicien moderne, comme on pourra les trouver dans les livres d'Algèbre et de Topologie de N. Bourbaki [cf. MR **3**, 55; **5**, 102; **6**, 113; **9**, 261, 406; **10**, 260, 261; **12**, 6, 40; **14**, 237, 1106], par exemple, auquels il convient d'ajouter expressément le livre de C. Chevalley "Theory of Lie groups" [Princeton, 1946; MR **7**, 412]."

These promises are fulfilled; and it takes only about 200 typewritten pages. In addition to the classical and recent literature (indicated in a carefully selected skeleton bibliography), unpublished manuscripts of R. Godement and N. Bourbaki have been used as sources. The material covered constitutes a natural continuation of Chevalley's "Theory of Lie Groups, I". A summary of the great bulk of results accounted for would necessarily become too long. Although the main results are already available in the literature, no such widely embracing coherent exposition of their totality has been made before. Here,

free use is made of all the relevant modern techniques and concepts in order to boil down the essentials of the theory to easily digestible proportions. In order to give at least an impression of the extent to which the field has been covered, we shall indicate some of the culminating points that are reached in this exposition.

About $\frac{3}{4}$th of the space is taken up by the theory of Lie algebras and their representations. With regard to the structure theory, this includes the complete classification of the simple Lie algebras over algebraically closed fields of characteristic 0 (which is done by the highly efficient method due to Dynkin), the Iwasawa-Mostow decomposition theorem for semisimple Lie algebras, over the field of the complex numbers, in combination with the result on the existence of compact real forms, and a complete analysis of the Cartan and Weyl groups (relative to a Cartan subalgebra) in their relation to the automorphism group of a semisimple Lie algebra. The account of the representation theory for semisimple Lie algebras includes the essentials of the theory of weights and characters, reaching well into the recent results of Harish-Chandra.

Among the results on the topological structure of Lie groups, one finds Iwasawa's decomposition theorem (reducing the problem to the case of compact groups), and also a very full account of the known results on the maximal toroidal subgroups of compact analytic groups, including the recent results of Borel and Serre [Comment. Math. Helv. **27** (1953), 128–139; MR **14**, 948]. With regard to the more classical results, we draw attention to the elegant proof of H. Weyl's theorem on the finiteness of the fundamental group of a compact semisimple analytic group. Actually, the following much more general theorem is proved: Let G be a connected topological group with a discrete normal subgroup C such that G/C is compact and coincides with the closure of its commutator group; then C is finite.

It is only natural that even this exposition cannot encompass everything; although the purely algebraic cohomology theory of Lie algebras is dealt with and applied to the elementary structure and representation theory, no room was found for establishing its connection with the topological cohomology theory, in the case of compact analytic groups.

G. Hochschild.

van Est, W. T. Une application d'une méthode de Cartan-Leray. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 542–544.

It is briefly indicated how earlier results of the author [same Proc. **56** (1953), 484–492, 493–504; **58** (1955), 225–233, 286–294; MR **15**, 505; **17**, 61] concerning the relations between the topological cohomology groups of a Lie group and the algebraic cohomology groups, computed from the group or its Lie algebra, in a representation space V can be obtained more simply from spectral sequences derived from suitable filtrations of bigraded complexes of (r, s)-cochains that are functions, with values in V, of r variables in the Lie algebra and $(s+1)$ variables in the group. For one of these spectral sequences, E_∞ is the cohomology group for the Lie algebra in V, while E_2 is the tensor product of the topological cohomology group of the group by the algebraic cohomology group of the group in V. The other spectral sequence concerns a (connected) Lie group M and an arbitrary compact subgroup F of M. Here E_∞ is the relative cohomology group for the Lie algebra of M, relative to that of F, in V, while E_2 is the tensor product of the topological cohomology group of the homogeneous space M/F by the algebraic cohomology group of M in V.

G. P. Hochschild (Berkeley, Calif.).

See also: Chevalley, p. 342; Toda, p. 395; Teleman, p. 404.

Topological Vector Spaces

Landsberg, Max. Pseudonormen in der Theorie der linearen topologischen Räume. Math. Nachr. **14** (1955), 29–38.

The author is chiefly concerned with topological vector spaces which are not locally convex. Neighborhoods of 0 in such a space E can be defined by equations $p(x)<1$, where p is a mapping from E into the positive real numbers, submitted to the condition $p(\lambda x)=|\lambda|p(x)$; in general p is not necessarily continuous for $x\neq 0$; the author shows that this property is equivalent to the fact that $p(x)=1$ is the frontier in E of the set $p(x)<1$. He next supposes that the dual E' of E separates points of E; then he shows that if E is a Baire space, its topology is finer than the Mackey topology $\tau(E, E')$. If E is metrizable, $\tau(E, E')$ is also metrizable, and is the coarsest metrizable topology for which the dual is E'; in the same case, a necessary and sufficient condition for E to be locally convex is that every weakly bounded subset of E be bounded for the initial topology of E. *J. Dieudonné.*

Dieudonné, Jean. Bounded sets in (F)-spaces. Proc. Amer. Math. Soc. **6** (1955), 729–731.

Two open questions in the theory of linear topological spaces are settled by example. Assuming the hypothesis of the continuum, the author exhibits: (A) a non-separable locally convex metrizable space, each bounded subset of which is separable, and (B) a bounded subset of the completion of a locally convex metrizable space E which is contained in the closure of no bounded subset of E. The question as to whether pathology of the type (A) can occur in a complete space remains open. *J. L. Kelley.*

Ghika, Al. Approximation des éléments d'un espace module normé général. Com. Acad. R. P. Române **1** (1951), 19–21. (Romanian. Russian and French summaries)

This review and the six following ones will use the concepts, notation, and terminology of five previous papers [Acad. R. P. Române, Bul. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 399–405, 509–517, 631–637, 653–661; Acad. R. P. Române. Stud. Cerc. Mat. **1**, 251–281 (1951); MR **13**, 565, 673; **16**, 715]. We shall refer to these papers as [1]–[5]. In the present paper a modular space (E, A) with respect to an F-ordered ring A is called total if for every singular $x\in E$ (i.e., such that there exists an idempotent divisor of zero ω with the property $\omega x=0$) there exists $x'\in E$ which is not singular such that $x=\omega x'$. The paper deals with the existence of nontrivial bounded linear functionals on a total (E, A) which are zero on prescribed subspaces of (E, A). *G. K. Kalisch.*

Ghika, Al. Propriétés des espaces modules normés généraux. Com. Acad. R. P. Române **1** (1951), 733–737. (Romanian. Russian and French summaries)

[See the note at the beginning of the preceding review.] After recalling several results of [3], the author generalizes portions of the results of Nachbin [Trans. Amer. Math. Soc. **68** (1950), 28–46; MR **11**, 369] to the spaces mentioned in the title of this paper. *G. K. Kalisch.*

Ghika, Al. Sur les anneaux F-ordonnés. Com. Acad. R. P. Române **2** (1952), 329–332. (Romanian. Russian and French summaries)

[See the note at the beginning of the 2nd preceding review.] The author proves that the second condition in the definition of an F-ordered ring in [1] is a consequence of the remaining conditions. *G. K. Kalisch.*

Ghika, Al. Continuité des applications linéaires dans les A-modules A-semi-normés. Com. Acad. R. P. Române **2** (1952), 405–408. (Romanian. Russian and French summaries)

[See the note at the beginning of the 3rd preceding review; see also Ghika, Acad. R. P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 563–583; MR **15**, 680.] Continuity of linear mappings between linear spaces of the type mentioned in the title is shown to be equivalent to the appropriate concept of boundedness.

G. K. Kalisch (Princeton, N.J.).

Ghika, Al. Sur la caractérisation des anneaux de fonctions réelles. Com. Acad. R. P. Române **2** (1952), 485–488. (Romanian. Russian and French summaries)

[See the note at the beginning of the 4th preceding review.] This paper contains an example showing that condition V of the representation theorem in [1] and [5] cannot be dispensed with even in the presence of the remaining conditions. *G. K. Kalisch.*

Ghika, Al. Ensembles A-convexes dans des Δ-modules. Com. Acad. R. P. Române **2** (1952), 669–671. (Romanian. Russian and French summaries)

[See the note at the beginning of the 5th preceding review.] This paper contains a generalization of the classical concept of convexity to Δ-modules where Δ contains a subring A whose additive group is a partially ordered group. Several of the usual consequences of this definition are stated. *G. K. Kalisch* (Princeton, N.J.).

Ghika, Al. Modules topologiques A-localement convexes. Com. Acad. R. P. Române **3** (1953), 101–103. (Romanian. Russian and French summaries)

[See the note at the beginning of the 6th preceding review.] On the basis of the concept of convexity introduced in the paper reviewed above, the author defines a corresponding concept of local convexity and states some necessary and sufficient conditions for local convexity of topological Δ-modules. *G. K. Kalisch.*

Ghika, Al. Modules topologiques localement convexes. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 49–73 (1953). (Romanian. Russian and French summaries)

This paper uses and elaborates concepts introduced by the author in earlier papers [see MR **13**, 565, 673; **15**, 680; **16**, 715; and the papers reviewed above] and applies them to generalize and analogize definitions given by J. Dieudonné [Ann. Sci. Ecole Norm. Sup. (3) **59** (1942), 107–139, quoted as D.; MR **6**, 178]. After establishing various needed consequences of his definitions, the author proves that a necessary and sufficient condition that a unitary Δ-module E be A-locally convex is that its topology be the sup of the topologies defined by a family of A-seminorms (with an obvious addition if E is to be separated). Here Δ is a ring which among other things contains an F-ordered subring A; A-local convexity means that E has a fundamental set of neighborhoods V of zero consisting of A-convex sets (i.e., a and b positive in A imply that $aV+bV\subset(a+b)V$) which are also A-centered and A-equilibrated (the latter are analogues of concepts in D.).

G. K. Kalisch (Princeton, N.J.).

Marinescu, G. L'inversion des polynômes différentiels et la division par des polynômes dans l'espace des distributions tempérées. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 297–300. (Romanian. Russiand and French summaries)

This note applies the results of a previous note [same Bul. **7** (1955), 65–74; MR **17**, 175; for notation, see the review cited]. Let P be a polynomial. The author claims that for all $S\in\mathcal{S}^*$, the equation $PT=S$ has a solution $T\in\mathcal{S}^*$. The proof is based on a misreading of a statement by L. Schwartz (see line 25, p. 298, where the proof fails).

E. Hewitt (Princeton, N.J.).

See also: Schieferdecker, p. 346; König, p. 354; Horváth, p. 362.

Banach Spaces, Banach Algebras

Fan, Ky, and Glicksberg, Irving. Fully convex normed linear spaces. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 947–953.

Let X be a normed, real-linear space. A sequence $\{x_n\}$ in X is called a $(k, 1)$-sequence if $\lim_n \|x_n\|=1$ and $\|\sum_{i\le k} x_{n_i}\|/k\ge 1$ for all $n_1\le\cdots\le n_k$. Other conditions (k, i), $i=2, 3, 4$, are defined and X is called $(F.k.C.i)$ $[(W.F.k.C.i)]$ if each (k, i)-sequence is Cauchy in norm [weak] topology. § 1 shows that for each k and for $1\le i, j\le 4$ the conditions $(F.k.C.i)$ and $(F.k.C.j)$ are equivalent; $(F.k.C.i)$ implies $(F.k+1.C.i)$ for each $k\ge 2$; for each $k\ge 2$ uniform convexity [Clarkson. Trans. Amer. Math. Soc. **40** (1936), 396–414] implies $(F.k.C.i)$ implies $(W.F.k.C.i)$ implies strict convexity. § 2 shows that for each $k\ge 2$ and $p>1$ an l^p product of $(F.k.C.i)$ spaces is $(F.k.C.i)$; an example of the reviewer [Bull. Amer. Math. Soc. **47** (1941), 313–317; MR **2**, 221] showed that such a space need not be uniformly convex even if the factors are; this shows that $(F.k.C.i)$ is not equivalent to uniform convexity. § 3 shows that X is reflexive if and only if every $(k, 3)$ sequence has a weak cluster point; hence every $(F.k.C.i)$ Banach space is reflexive and every $(W.F.k.C.i)$, weakly sequentially complete space is reflexive. § 4 shows that if X has a separable adjiont space, then Clarkson's renorming of X is $(W.F.k.C.i)$ for every $k\ge 2$. § 5 is an ergodic theorem for sequences in a $(W.F.k.C.i)$ space.

M. M. Day (Urbana, Ill.).

Matsushita, Shin-ichi. Positive functionals and representation theory on Banach algebras. I. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 1–18.

This paper supplies proofs for the results announced previously [Proc. Japan Acad. **29** (1953), 427–430; MR **16**, 375]. *J. G. Wendel* (Ann Arbor, Mich.).

Smiley, M. F. Right annihilator algebras. Proc. Amer. Math. Soc. **6** (1955), 698–701.

Given a Banach space X, let $F(X)$ denote the algebra of all approximately finite-valued operators in X. It is proved that a topologically simple Banach algebra A is isomorphic and isometric to some $F(X)$ if and only if every left ideal which is not dense in A has a non-zero

right annihilator and the norm in A satisfies the following condition. If $\|a\|=1$ and $0<\varepsilon<1$, then for some element b in A with $\|b\|=1$, we have $\|(ab)^n\|\geq(1-\varepsilon)^n$ for every positive integer n. A characterization of this kind was previously known for reflexive Banach spaces X. It is further proved that if A satisfies the above conditions but is perhaps not topologically simple, then it is the completion of the direct join of its minimal closed two-sided ideals, each of which is isomorphic and isometric to an $F(X)$. *F. F. Bonsall* (Newcastle-upon-Tyne).

Edwards, R. E. On certain algebras of measures. Pacific J. Math. **5** (1955), 379–389.

L'auteur montre que la "synthèse harmonique" est possible pour l'espace des fonctions continues à croissante lente sur la droite réelle R, muni de la topologie définie par la famille de semi-normes

$$p_\sigma(f)=\sum_{n=-\infty}^{+\infty}\sigma_n\sup_{n-1\leq x\leq n}|f(x)|,$$

où σ parcourt l'ensemble des suites $(\sigma_n)_{n=-\infty}^{\infty}$ à décroissance rapide. Il montre que celà résulte aussitôt par régularisation d'un résultat analogue du reviewer relatif à l'espace $\mathcal{S}'$ des distributions tempérées [Canad. J. Math. 3 (1951), 503–512; MR 13, 473]. *L. Schwartz* (Paris).

★Nakano, Hidegorô. Semi-ordered linear spaces. Japan Society for the Promotion of Science, Tokyo, 1955. vi+508 pp. $6.00.

This is a reprint by photo-offset of twenty-one papers by the author (in one case a joint paper with I. Halperin) on partially ordered linear spaces. They have all been reviewed: MR 3, 210; 7, 249, 453 (2d paper); 8, 387; 9, 191, 445 (1st paper); 10, 550; 13, 49, 954; 15, 137, 442; 16, 49. The author states in a preface that he felt impelled to give wider circulation to these papers since many of the results, particularly of the earlier papers, were being rediscovered by others.

Yamamuro, Sadayuki. Modulared sequence spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 1–12.

For each index ν suppose given a function $f_\nu(u)$ with; $0\leq f_\nu(u)\leq\infty$ for $0\leq u\leq\infty$; $f_\nu(0)=0$; $f_\nu(u)\to\infty$ as $u\to\infty$; $f_\nu(u)$ not identically infinite for all $u>0$; and f_ν non-decreasing, convex. For each sequence $x=(\xi_\nu)$ define $m(x)=\sum_\nu f_\nu(|\xi_\nu|)$. Let $l(f_\nu)$ denote the set of x with $m(\alpha x)<\infty$ for some $\alpha>0$. Then $l(f_\nu)$ is a modulared sequence space with m as modular; it is a (complete) Banach space with norm $\|x\|=\inf\eta$ for all $\eta>0$ with $m(x/\eta)\leq 1$. Necessary and sufficient conditions are given for the properties (i) $l(f_\nu)$ contains l_1 (the summable sequences) and (ii) $l(f_\nu)$ contains only bounded sequences.

Let $\bar{f}_\nu$ denote the complementary function of f_ν, in the sense of Young, i.e. f_ν' and $\bar{f}_\nu'$ are inverse functions. The author shows that $l(\bar{f}_\nu)$ is the conjugate modular space of $l(f_\nu)$, with conjugate modular $\bar{m}=\sum_\nu \bar{f}_\nu$. For two such spaces, it is shown that $l(f_\nu)\subset l(g_\nu)$ if and only if, for some $\alpha>0$, $\sum_\nu f_\nu g_\nu^{-1}(\alpha)<\infty$; here g_ν^{-1} is the inverse function of g_ν and the very restrictive assumption is made that $f_\nu g_\nu^{-1}$ is convex.

Schur's property, that weak and strong convergence are equivalent in $l(f_\nu)$, is shown to hold if: $\chi'_\nu\to 1$ and only if: $\chi_{f_\nu}\to 1$, but with assumed restrictions on the f_ν. Here χ'_ν (χ_{f_ν}) is the greatest lower (least upper) bound of the $p\geq 1$ for which $f_\nu(u)/u^p$ is non-increasing (non-decreasing). *I. Halperin* (Kingston, Ont.).

Yamamuro, Sadayuki. On finite modulars. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 13–21.

Let m be a Nakano modular on a partially ordered vector space R. This m is called finite if $m(x)<\infty$ for all x and upper bounded if $m(\alpha x)\leq\gamma m(x)$ for all x, for some fixed $\alpha>1$, $\gamma<\infty$. Finiteness is implied by upper boundedness but the converse need not hold. However, the author shows that finiteness does imply a weakened form of upper boundedness. If R is, in particular, a modulared sequence space [see the preceding review], then this result is strengthened to generalize a theorem which Orlicz and Birnbaum obtained for the case that all f_ν coincide. *I. Halperin* (Kingston, Ont.).

Amemiya, Ichiro. A characterization of the modulars of L_p type. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1954), 22–33.

Let $m(x)$ be a modular (in the sense of H. Nakano) on a conditionally complete vector lattice R. Then m gives rise to two norms: $\|x\|=\inf(1+m(\xi x))/\xi$ for all $\xi>0$ and $|||x|||=\inf\eta$ for all $\eta>0$ with $m(x/\eta)\leq 1$. In particular, if R is L_p ($p\geq 1$) and $m(x)=\int|x(t)|^p$, then $\|x\|=\alpha|||x|||$ for α a constant. The author proposes to prove that, conversely, proportionality of the two norms, with $\alpha>1$, does imply that R is a space L_p (except for a slight variation). He assumes that R is infinite dimensional; but his proof that "$|||x|||=1$ implies $m(x)=1$" seems to assume that R has no discrete elements. In § 2, from $m(x_\nu)=\frac{1}{2}$ for orthogonal x_ν, the author derives that $m(2x_\nu)=1$; but why $m(2n^{-1}(x_1+\cdots+x_n))=1$ holds (as stated) is not clear to the reviewer. *I. Halperin* (Kingston, Ont.).

Ghika, A. Beschränkte Linearformen. Acad. R. P. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 693–697. (Romanian. Russian and French summaries)

On établit certaines propriétés fondamentales des formes linéaires bornées (fontionnelles générales linéaires). Théorème 1. Etant donné une forme linéaire bornée f définie sur un sous-espace module G d'un espace module semi-normé (E, A) [v. Ghika, Acad. R.P. Române, Bul. Şti., Ser. Mat. Fiz. Chim. **2** (1950), 399–405; MR **13**, 565] par rapport à un anneau F-ordonné A, il existe une forme linéaire bornée g définie sur E telle que $f(x)=g(x)$ pour tout $x\in G$ et telle que $\|g\|=\|f\|_G$ [où $\|g\|=\sup_{\|x\|\leq 1}|g(x)|$]. Théorème 2. Pour tout élément x_0 appartenant à (E, A) il existe une forme lineaire bornée $f_0(x)$ définie sur E telle que $f_0(x_0)=\|x_0\|$ et $\|f_0\|=\varepsilon$. Théorème 3. Soit $h(x)$ une application d'une partie H d'un espace module semi-normé (E, A) dans A et μ un élément de A strictement positif. Pour qu'il existe une forme linéaire bornée f définie sur E telle que $f(x)=h(x)$ pour tout $x\in H$, et $\|f\|\leq\mu$ il faut et il suffit que $|\sum_{i=1}^n\alpha_i h(x_i)|\leq\mu\|\sum_{i=1}^n\alpha_i x_i\|$ pour toute suite finie $(x_i)_{1\leq i\leq n}$ d'éléments de H et pour toute suite finie $(\alpha_i)_{1\leq i\leq n}$ d'éléments de A. *Autoreferat* (Zbl **41**, 437).

Slugin, S. N. Approximate solution of operator equations on the basis of S. A. Čaplygin's method. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 565–568, 746. (Russian)

The methods of S. A. Čaplygin for the approximation of solutions to differential and difference-differential equations can be extended to the solution of operator equations on partially ordered linear spaces. The solutions are approximated by means of upper and lower successive approximations which converge monotonically. Such an extension was made by A. N. Baluev [Dokl. Akad. Nauk (N.S.) **83** (1952), 781–784; MR **14**, 185], but the author states that it is not possible to derive Čaplygin's algorithm

from Baluev's theory. This paper contains the statement of two theorems, the first of which contains numerous conditions and is lengthy to state. The second theorem is a special case of the first, is somewhat simpler to state, and illustrates the general type of result under discussion.

Let P be an operator defined on an interval $[\underline{x}_0, \bar{x}_0]$ of a partially ordered linear space. Let Γ be an additive operator having a positive inverse Γ^{-1} and such that for all $\Delta x>0$, $\Gamma(\Delta x)\geqq P(x+\Delta x)-P(x)$. If the operator $\Gamma^{-1}P$ is monotonic and continuous and $P(\underline{x}_0)\leqq 0\leqq P(\bar{x}_0)$, then the least and greatest solutions in $[\underline{x}_0, \bar{x}_0]$, denoted by $\underline{x}$ and $\bar{x}$, respectively, of the equation $P(x)=0$ can be obtained by the iterative procedure

$$\bar{x}_{n+1}=\bar{x}_n-\Gamma^{-1}P(\bar{x}_n),\ \underline{x}_{n+1}=\underline{x}_n-\Gamma^{-1}P(\underline{x}_n),$$

and

$$\bar{x}_n\searrow\bar{x},\ \underline{x}_n\nearrow\underline{x},\ \text{as } n\to\infty.$$

Sufficient conditions are given for the uniqueness of the solutions. It is shown that Čaplygin's algorithm and two of Baluev's theorems can be deduced from these theorems.

J. P. LaSalle (Notre Dame, Ind.).

See also: Schieferdecker, p. 346; Calderón, p. 362; Davis, p. 389; Ghika, p. 402.

Hilbert Space

Putnam, C. R. On rotations in Hilbert space. Quart. J. Math. Oxford Ser. (2) **6** (1955), 188–192.

In a real Hilbert space of dimension $\aleph_0$ the author investigates the structure of the group of unitary transformations in the uniform operator topology. It is shown that the subset of rotations (elements of the form e^s for some skew-symmetric bounded linear operator s) is locally arcwise connected. *J. Feldman* (Princeton, N.J.).

Zuhovickiĭ, S. I. Some theorems of the theory of Čebyšev approximations in Hilbert space. Mat. Sb. N.S. **37 (79)** (1955), 3–20. (Russian)

Let Q be a compact metric space and H a complex Hilbert space. Let F_k $(k=1, 2, \cdots, n)$ be mappings of Q into the ring of bounded linear operators on H such that $\|F_k(q)\|$ depends continuously on $q\in Q$. Let Φ be a continuous mapping of Q into H (norm topology). The approximation problem studied here is that of finding elements $A_1, A_2, \cdots, A_n\in H$ for which

$$\max\{\|\sum_{k=1}^{n}F_k(q)A_k-\Phi(q)\|:q\in Q\}$$

attains its minimum. Theorem: Let $n=1$. Let R be the subspace of all $A\in H$ such that $F_1(q)A=0$ for all $q\in Q$. Suppose that there is a positive real number m such that $m\|A\|\leqq\max\{\|F_1(q)A\|:q\in Q\}$ for all $A\in R^{\perp}$. Then there is an element $A\in H$ for which $\max\{\|F_1(q)A-\Phi(q)\|:q\in Q\}$ attains its minimum value. Conditions for uniqueness of A are given. An existence theorem for the general case, very like that for $n=1$, is also given. *E. Hewitt.*

Zuhovic'kiĭ, S. I. On the problem of Čebyšev approximation in Hilbert space. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 7–11. (Ukrainian. Russian summary)

Excerpts from the paper reviewed above.

E. Hewitt (Princeton, N.J.).

Meschkowski, Herbert. Über Hilbertsche Räume mit Kernfunktion. (Berichtigung). Arch. Math. **6** (1955), 481.

See same vol., 151–156; MR **16**, 596.

Rapoport, I. M. On estimation of eigenvalues of Hermitian operators. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 199–202. (Russian)

Let $\cdots<\lambda_{k-1}<\lambda_k<\lambda_{k+1}<\cdots$ be part of the discrete spectrum of a Hermitian linear operator A on a Hilbert space H. The author finds inequalities for λ_k, presupposing knowledge of an upper bound $\bar{\lambda}_{k-1}$ for λ_{k-1}, a lower bound $\underline{\lambda}_{k+1}$ for λ_{k+1}, and a normalised $x\in H$ such that $\bar{\lambda}_{k-1}+\|\delta\|\leqq(Ax, x)\leqq\underline{\lambda}_{k+1}-\|\delta\|$, where $\delta=Ax-(Ax, x)x$. In fact

$$(Ax, x)-\|\delta\|^2(\underline{\lambda}_{k+1}-(Ax, x))^{-1}\leqq\lambda_k\leqq$$
$$\leqq(Ax, x)+\|\delta\|^2((Ax, x)-\bar{\lambda}_{k-1})^{-1}.$$

This is specialised for the $2n$th-order Sturm-Liouville case. In a numerical application to a second-order Sturm-Liouville problem the method appears highly effective.

F. V. Atkinson (Canberra).

Mautner, F. I. On expansions in eigenfunctions. Uspehi Mat Nauk (N.S.) **10** (1955), no. 4(66), 127–132. (Russian)

Translation of Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 49–53; MR **14**, 659.

Gel'fand, I. M., and Kostyučenko, A. G. Expansion in eigenfunctions of differential and other operators. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 349–352. (Russian)

Notations: Φ=espace de Fréchet de fonctions indéfiniment différentiables sur R^n; Φ'=dual de Φ; on suppose $\Phi\subset\Phi'$. On donne sur Φ une forme sesquilinéaire (φ, ψ) définie positive continue; soit $\overline{\Phi}$ le complété de Φ pour cette structure; on a: $\Phi\subset\Phi\subset\Phi'$. Soit A opérateur linéaire continu de Φ dans Φ avec $(A\varphi, \psi)=(\varphi, A\psi)$ pour tout φ, $\psi\in\Phi$. On suppose qu'il existe $\bar{A}$, hypermaximal dans $\overline{\Phi}$, d'ensemble de définition contenant Φ, avec $\bar{A}\varphi=A\varphi$ pour tout $\varphi\in\Phi$. Soit A' le transposé de A dans Φ'. Un élément T de Φ' est dit "fonction propre généralisée" si $A'T=\lambda T$, $\lambda\in C$. Théorème: Sous les hypothèses ci dessus, il existe un système complet de fonctions propres généralisées.

Les A. en déduisent un résultat de „diagonalisation" de A [cf. des résultats plus précis dans les travaux de Mautner et Gârding; cf. notamment Gârding, Applications of the theory of direct integrals of Hilbert spaces to some integral and differential operators, Univ. of Maryland, 1954 [MR **17**, 159] où l'on trouvera d'autres indications bibliographiques]. Exemples: A est un opérateur différentiel sur R ou sur R^n, elliptique ou non; certains espaces Φ sont construits à partir des propriétés de croissance des coefficients de A. Généralisation à $A+\lambda B$, $B>0$.

J. L. Lions (Nancy).

Brownell, F. H. Fourier analysis and differentiation over real separable Hilbert space. Pacific J. Math. **5** (1955), 649–662.

The subset X_0 of real sequence Hilbert Space l_2, where $X_0=\{x: x\in l_2, |x_n|<h_n \text{ for } n\geqq N\}$ where h_n is some fixed sequence of numbers such that $\sum h_n^2$ is finite, is made into a topological group by adding the nth components modulo $2h_n$, and imposing a topology induced by that of l_2.

The character group is the set Z_0 in l_2, with $z_n=\pi p_n/h_n$, p_n integral, for $n\geqq N$, the continuous characters having the form $\exp i(z, x)$. Fourier transforms in terms of the Haar measure on these groups are discussed, and it is shown that differentiability with respect to the nth coordinate x_n, with differential coefficient and function in the space of summable or summable-square functions over X_0, implies that the derivative has Fourier transform $-iz_n$ times that of the function. A Laplacian with Fourier transform $(-\sum z_n^2)F$, where F is the transform of the original function, is defined and compared with Gateaux and Fréchet-type definitions. *J. L. B. Cooper.*

Davis, Chandler. Generators of the ring of bounded operators. Proc. Amer. Math. Soc. **6** (1955), 907–972.

The ring generated by operators A, B, $\cdots$, on a separable Hilbert space $\mathcal{H}$ is defined to be the minimal weakly closed self-adjoint algebra containing A, B, $\cdots$ and the identity operator. $\mathcal{B}$ is the ring of all bounded operators on $\mathcal{H}$. Theorem 1: $\mathcal{B}$ can be generated by three projections, and by no smaller number if dim $\mathcal{H}\geqq 3$. Theorem 2: $\mathcal{B}$ can be generated by two unitary operators, one of which is a symmetry. *J. H. Williamson.*

Sakai, Shôichirô. On the group isomorphism of unitary groups in AW-algebras. Tôhoku Math. J. (2) **7** (1955), 87–95.

It is shown that if M and N are AW^* factors, M_u, N_u their respective unitary groups, and ϱ a uniformly continuous group isomorphism from M_u into N_u, then there is a unique map f from M onto N which is either a linear or conjugate linear * isomorphism and which agrees with ϱ on M_u. This has been proven by H. A. Dye for W^* factors [Ann. of Math. (2) **61** (1955), 73–89; MR **16**, 598], by a different method which also gives results when no continuity assumption is made on ϱ. *J. Feldman.*

Nakamura, Masahiro, and Turumaru, Takasi. On extensions of pure states of an abelian operator algebra. Tôhoku Math. J. (2) **6** (1954), 253–257.

Soient A une C^*-algèbre abélienne dans un espace hilbertien H, π un état pur de A. Les AA. disent que π est du type onde (of wave type) si $\pi(x)=\langle x\varphi, \varphi\rangle$ pour un $\varphi\in H$, et que π est de type idéal si π se prolonge en un état pur de $\mathcal{L}(H)$ (algèbre de tous les opérateurs de H) nul sur les opérateurs complètement continus. Exemple de résultat: si π est du type idéal et si x est un élément hermitien de A, $\pi(x)$ appartient au spectre condensé de x. § 3: correction à une démonstration antérieure [Proc. Japan Acad. **29** (1953), 501–502; MR **15**, 968] d'un théorème de Rosenberg. § 4: généralisation d'un théorème d'Ogasawara [J. Sci. Hiroshima Univ. Ser. A. **17** (1951), 359–364; MR **16**, 598] sur la caractérisation des opérateurs complètement continus dans l'algèbre des opérateurs d'un espace hilbertien. *J. Dixmier* (Paris).

See also: Kay and Moses, p. 372; Nakano, p. 387.

TOPOLOGY

Iséki, Kiyoshi. On the property of Lebesgue in uniform spaces. I, II, III, IV, V. Proc. Japan Acad. **31** (1955), 220–221, 270–271, 441–442, 524–525, 618–619.

Kasahara, Shouro. On the Lebesgue property in uniform spaces. Math. Japon. **3** (1955), 127–132.

Kasahara, Shouro, and Kasahara, Kouji. Note on the Lebesgue property in uniform spaces. Proc. Japan Acad. **31** (1955), 615–617.

Since these papers all deal with the same topic, and since the later ones are partly devoted to improving results obtained in earlier ones, they will all be reviewed together. The five Iséki papers will be referred to by I1–I5, and the two Kasahara papers by K1 and K2.

Let X be a Hausdorff uniform space, with uniform structure $\mathcal{U}$. According to [I1], a covering (=open covering) $\mathcal{U}$ of X has the Lebesgue property if there exists a vicinity $V\in\mathcal{U}$ such that, for every $x\in X$, $V(x)\subset U$ for some $U\in\mathcal{U}$. [For metrizable $(X, \mathcal{U})$, this concept was studied by Monteiro and Peixoto, Portugal. Math. **10** (1951), 105–113; MR **13**, 447.] Most of the authors' results can be summarized in the following four theorems: A [I4, Th. 3]: If every binary covering of X has the Lebesgue property, then every finite covering has it. B [I4, Th. 4]: Every finite covering of X has the Lebesgue property if and only if X is normal and every bounded continuous real-valued function on X is uniformly continuous. C[K2, Cor. 2 to Th. 1]: Every covering of X has the Lebesgue property if and only if X is paracompact and $\mathcal{U}$ is the universal (=finest) uniform structure on X. D[K1, Cor. 1 to Th. 6]: If $(X, \mathcal{U})$ is metrizable, and if every covering of X has the Lebesgue property, then X is the union of a compact subset and a discrete subset. [Reviewer's note: In [I5], the theorem that every paracompact space is complete in its universal uniform structure is credited to A. Dickinson [Amer. J. Math. **75** (1953), 224–228; MR **14**, 1107]; the first proof, however, was given by J. Nagata [J. Math. Soc. Japan **2** (1950), 44–47; MR **12**, 626].] *E. Michael.*

Kunugi, Kinjirô. Sur les espaces complets et régulièrement complets. I. Proc. Japan Acad. **30** (1954), 553–556.

L'A. définit, dans le cas d'espaces topologiques très généraux, les notions de profondeur de l'espace (notion analogue bien que non équivalente à celle de caractère), et d'espace rangé consistant dans l'attribution à certains voisinages d'un ordinal appelé rang de ce voisinage et vérifiant certaines conditions. L'A. généralise aux espaces rangés les notions de suite décroissante fondamentale (transfinie) de voisinages et d'espace complet; il démontre dans les espaces rangés complets le théorème de Baire. Il introduit aussi une notion d'espace rangé régulièrement complet. *A. Appert* (Angers).

Kunugi, Kinjirô. Sur les espaces complets et régulièrement complets. II. Proc. Japan Acad. **30** (1954), 912–916.

Tout espace localement bicompact peut être considéré comme un espace rangé complet. Il existe des espaces rangés complets qui sont partout irréguliers et donc non uniformisables. L'axiome de séparation de Hausdorff pour les espaces rangés n'implique pas la régularité de l'espace (contrairement à ce qui se passe pour les espaces uniformes). L'A. traite le problème de la complétion d'un espace rangé par un procédé essentiellement différent de celui employé par A. Weil pour les espaces uniformes. Le procédé de complétion d'A. Weil est inapplicable à un espace rangé partout irrégulier. *A. Appert* (Angers).

Kunugi, Kinjirô. Sur les espaces complets et régulièrement complets. III. Proc. Japan Acad. **31** (1955), 49–53.

Suite de l'étude du problème de la complétion d'un espace rangé. *A. Appert* (Angers).

Kowalsky, Hans-Joachim. Limesräume und Komplettierung. Math. Nachr. **12** (1954), 301–340.

Le concept des espaces $\mathcal{L}$ de M. Fréchet permettait d'embrasser les cas de certains espaces topologiques non transitifs, c'est-à-dire ne vérifiant pas l'axiome $\overline{\overline{E}}=\overline{E}$ ($\overline{E}$ désignant la fermeture d'un ensemble quelconque E). Mais ce concept d'espace $\mathcal{L}$, étant lié à l'idée de suite infinie dénombrable de points, était par le fait même soumis à des restrictions de dénombrabilité. L'A. s'affranchit de ces restrictions en substituant les filtres dans l'espace aux suites infinies.

L'A. appelle espace-limite un système (R, T) constitué par un ensemble quelconque R et une fonction Tx faisant correspondre à tout $x \in R$ une famille Tx de filtres dans R vérifiant certaines conditions. Les filtres $\in Tx$ sont appelés les filtres convergeants vers x. Ils induisent dans R une topologie (généralisée).

Les définitions de l'A. sont suffisamment générales pour que tout espace topologique (généralisée) vérifiant les seuls axiomes:

$$E \subset \overline{E},\ \overline{0}=0,\ \overline{E \cup F}=\overline{E}\cup\overline{F},$$

puisse être considéré comme un espace-limite et ceci en général de plusieurs manières.

Conditions nécessaires et suffisants pour qu'un espace-limite vérifie l'axiome de transitivité $\overline{\overline{E}}=\overline{E}$ et l'un ou l'autre des axiomes topologiques de séparation. Applications aux groupes, aux espaces uniformes et à divers problèmes de complétion. *A. Appert* (Angers).

Nöbeling, Georg, und Bauer, Heinz. Über die Erweiterungen topologischer Räume. Math. Ann. **130** (1955), 20–45.

Let E be a topological space (no separation axioms assumed). Let $\mathfrak{C}E$ be the real Banach algebra of all real-valued bounded continuous functions on E. The basic facts about $\mathfrak{C}E$ were obtained by M. H. Stone [Trans. Amer. Math. Soc. **41** (1937), 375–481]. Additional information was gained by I. M. Gel'fand [Mat. Sb. N.S. **9(51)** (1941), 3–24; MR **3**, 51] and by I. M. Gel'fand and G. I. Šilov [ibid. **9(51)** (1941), 25–39; MR **3**, 52]. For the case in which E is a completely regular T_0-space, the properties of $\mathfrak{C}E$ relevant to the present paper are as follows. Given a closed subalgebra $\mathfrak{A}$ of $\mathfrak{C}E$ containing 1, there is a unique compact Hausdorff space F such that $\mathfrak{A}$ and $\mathfrak{C}F$ are isomorphs. F contains a continuous image E^* of E as a dense subspace. If $\mathfrak{A}$ separates points in E, then E^* is a 1-to-1 image of E, and if $\mathfrak{A}$ separates points and closed sets, E^* is a homeomorph of E.

By eliminating separation axioms and broadening the notion of extension of a topological space, the authors generalize these results and in fact obtain all standard compactifications as quite special cases. Let E be a topological space with a closure operator h and let E' be a superset of E that is a topological space with closure operator h'. If $hA \subset h'A$ for all $A \subset E$ (i.e., if the topology of E is no weaker than the relative topology of $E \subset E'$), then E' is called an extension of E. Note that E need not be dense in E'. Theorem: Let E, S be topological spaces and let λ be a homomorphism of $\mathfrak{C}S$ into $\mathfrak{C}E$. Then there is an extension εE of E such that $\mathfrak{C}(\in E)$ and $\mathfrak{C}S$ are isomorphs and the mapping $f \to f|E$ for $f \in \mathfrak{C}(\in E)$ reproduces the homomorphism λ.

The family of all extensions of a given topological space E is now divided into classes. Two extensions E' and E'' of E lie in the same extension class $\mathfrak{E}E$ if there is an isomorphism Φ of $\mathfrak{C}E'$ onto $\mathfrak{C}E''$ such that $f'|E=(\Phi f')|E$ for all $f' \in \mathfrak{C}E'$. Theorem: Every extension class $\mathfrak{E}E$ contains a space $\beta_{\mathfrak{E}}E$ that is compact, completely regular (usually not T_0!), and for which every point of $\beta_{\mathfrak{E}}E-E$ is closed. The space $\beta_{\mathfrak{E}}E$ is unique up to E-identical homeomorphisms. Theorem: Every extension class $\mathfrak{E}E$ contains an extension E' such that E has the relative topology in E', E' is compact, and $E'-E$ is closed, normal, and T_1. Conditions are given under which the extension over $E' \in \mathfrak{E}E$ of a given $f \in \mathfrak{C}E$ is possible in at most one way: when this occurs, $\mathfrak{E}E$ is called a class of unique extensibility. Theorem: If $\mathfrak{A}$ is any closed subalgebra of $\mathfrak{C}E$ containing 1, then there is a unique class $\mathfrak{E}E$ of unique extensibility such that the functions in $\mathfrak{A}$ are exactly the functions admitting continuous extensions over all $E' \in \mathfrak{E}E$.

Various special extensions of E are studied. There exists a unique compact extension $\beta_{\mathfrak{C}}E$ of E such that all $f \in \mathfrak{C}E$ can be extended over $\beta_{\mathfrak{C}}E$, $\beta_{\mathfrak{C}}E$ is completely regular, and all points of $\beta_{\mathfrak{C}}E-E$ are closed. If E is completely regular and T_0, then $\beta_{\mathfrak{C}}E$ is the classical Stone-Čech compactification of E. The compactifications of Katětov [Časopis Pěst. Mat. Fys. **72** (1947), 17–32; MR **9**, 153], Aleksandrov [Mat. Sb. N.S. **5(47)** (1939), 403–423; MR **1**, 318], and Wallman [Ann. of Math. (2) **39** (1938), 112–126] are also obtained. The present paper thus unifies in an elegant way constructions that previously looked very different. *E. Hewitt.*

Schaefer, Helmut. Stetige Konvergenz in allgemeinen topologischen Räumen. Arch. Math. **6** (1955), 423–427.

In this paper the concept of continuous convergence introduced by H. Hahn [Theorie der reellen Funktionen, Springer, Bd. 1, Berlin, 1921, pp. 238–246] is generalized to sequences of functions from an arbitrary topological space X into a complete metric space Y. Let $\{f_n\}$ be a sequence of arbitrary functions from X into Y. For any point $x_0 \in X$, let $F(x_0; \{f_n\})$ denote the filter on Y generated by the family of all sets of the form $S(n, V)=\bigcup_{i \geq n} f_i(V)$, where $n=1, 2, 3, \cdots$ and V runs through all neighborhoods of x_0 in X. The sequence $\{f_n\}$ is said to converge continuously at x_0, if the filter $F(x_0; \{f_n\})$ converges to a point of Y. It is shown that $\{f_n\}$ converges continuously at x_0, if and only if $\{f_n\}$ converges locally uniformly at x_0 and the oscillation of f_n at x_0 tends to 0 as $n \to \infty$. When X is locally compact, $\{f_n\}$ converges continuously at every point of X, if and only if $\{f_n\}$ converges uniformly on every compact subset of X and the limit function is continuous on X. The author also discusses the related concept of normal family of functions in the general setting. *Ky Fan* (Notre Dame, Ind.).

Moisil, Gr. C. Une définition des espaces uniformes. Com. Acad. R. P. Romîne **5** (1955), 909–912. (Romanian. Russian and French summaries)

As primitive idea to define a uniform space, the author considers the relation $W(a, b, c, d)$ between 4 points, meaning intuitively: "a and b are not farther from one another than c from d". He imposes on that relation a system of 7 axioms, which correspond to obvious con-

ditions the relation satisfies in a uniform space; and conversely he shows that if one defines $V_{(c,d)}$ to be the set of pairs (a, b) such that $W(a, b, c, d)$, these sets constitute a fundamental family of surroundings for a uniform structure. *J. Dieudonné* (Evanston, Ill.).

Geymonat, Ludovico. Sulle combinazione di due caratteristiche trasformazioni della contiguità negli spazi topologici. Rend. Mat. e Appl. (5) **14** (1954), 200–208.

Dans une précédente note [Rend. Mat. e Appl. (5) **12** (1954), 360–366; MR **16**, 387] l'A. avait associé à tout espace topologique S au sens de M. Fréchet, c'est-à-dire à tout espace défini par une opération de fermeture $\overline{E}=\varrho E$ vérifiant les deux axiomes $E\subset\varrho E$ et $\varrho 0=0$, un espace topologique $\dot{S}$ défini par l'opération de fermeture $\varrho^*E=$ complémentaire de $(\varrho E-E)$ dite conjuguée de ϱ(on convient de plus que $\varrho^*0=0$). L'espace $\dot{S}$ est dit le conjugué de S.

Dans le présent mémoire, l'A. envisage [à la suite de A. Appert, Mathematica, Cluj **11** (1935), 229–246, p. 245] l'espace $\tilde{S}$ que j'avais appelé l'espace $(\mathcal{U})$ associé à S et où l'opération de fermeture est $\tilde{\varrho}E=\mathbf{U}_{e\in E}\,\varrho e$.

Il considère les deux propriétés:

$$(\alpha)\quad \tilde{\dot{S}}=S \qquad (\beta)\quad \dot{\tilde{S}}=S.$$

Il donne des exemples d'espaces topologiques vérifiant (α) sans vérifier (β), et d'espaces topologiques vérifiant (β) sans vérifier (α). La condition nécessaire et suffisante pour avoir simultanément (α) et (β) est que S et $\dot{S}$ soient tous deux des espaces $(\mathcal{U})$. *A. Appert* (Angers).

Geymonat, Ludovico. La spazializzazione degli insiemi. Univ. e Politec. Torino. Rend. Sem Mat. **13** (1954), 47–58.

Conférence de caractère philosophique et historique sur l'arithmétisation du continu spatial et la topologie abstraite ensembliste (il n'est pas question de la topologie combinatoire). L'A. y fait une place à ses propres recherches sur les espaces topologiques conjugués d'un espace topologique donné. *A. Appert* (Angers).

Bartle, R. G. Nets and filters in topology. Amer. Math. Monthly **62** (1955), 551–557.

This is a very lucid account of the precise relation between Moore-Smith convergence and the theory of filters. It includes a proof of the existence of universal subnets, based on the existence of ultrafilters.

J. L. Kelley (Berkeley, Calif.).

Dolcher, Mario. Topologia delle famiglie di filtri. Rend. Sem. Mat. Univ. Padova **24** (1955), 443–473.

Let S be a set; A, B non-empty subsets of S; $\alpha, \beta, \gamma, \varphi$ filters in S, and Γ, Φ, Θ families of filters. The filter of all super sets of A is denoted by $[A]$. For any Φ the author defines a strong and a weak topology on Φ. This process the author regards as analogous to the processes of compactification or completion. Relations $\bigcirc$ and $\triangle$ are defined for filters which are analogous to $A\cap B\neq\emptyset$ and $A\subset B$ for sets. 1) α is bound to β ($\alpha\bigcirc\beta$) if $A\in\alpha$ and $B\in\beta$ imply $A\cap B\neq\emptyset$. 2) Γ is bound to α ($\Gamma\bigcirc\alpha$ or $\alpha\bigcirc\Gamma$) if for every $A\in\alpha$ there exists a $\gamma\in\Gamma$ such that $\gamma\bigcirc[A]$. 3) Γ occurs in α ($\Gamma\triangle\alpha$) if for every $A\in\alpha$ there exists a $\gamma\in\Gamma$ such that $A\in\gamma$.

The strong topology in a family Φ is defined by the closure operator $\Theta'=\{\varphi\in\Phi\mid\Theta\triangle\varphi\}$. With this topology Φ is a T_0 space. Conditions are given that the topology be T_1, Hausdorff, or regular. He proves that if S is a T_0 space and Σ is the family of all filters of neighborhoods of points of S and if $\Sigma\subset\Phi$ then S is homeomorphic to Σ as a subspace of Φ in the strong topology.

A set Γ is absolutely closed in Φ if it contains any filter bound to it. The closure Γ''' in the weak topology is defined to be the intersection of all the absolutely closed families containing Γ. If S is a topological space and Σ is as before, then S is homeomorphic to Σ in the weak topology if and only if S is regular. The theorem for $\Sigma\subset\Phi$ is false as an example shows. *M. E. Shanks* (West Lafayette, Ind.).

Rudin, Mary Ellen. Countable paracompactness and Souslin's problem. Canad. J. Math. **7** (1955), 543–547.

Dowker [same J. **3** (1951), 219–224; MR **13**, 264] raised the question: is every normal Hausdorff space T countably paracompact (i.e., does every countable open cover of T have a locally finite open refinement)?—this problem has already achieved a celebrated status. Souslin [Fund. Math. **1** (1920), 223] raised the question: for every (linearly) ordered set, does (1) every family of disjoint intervals is countable, imply (2) there exists a countable dense subset?—this is one of the famous unsolved problems of set theory. The result established in the paper under review is that the affirmative answer to Dowker's problem implies the affirmative answer to Souslin's problem.

The proof is replete with ingenious arguments. Suppose that there exists an ordered set satisfying (1) but not (2); then there is an interval s none of whose subintervals satisfies (2). A natural (and well-known) construction yields $\aleph_1$ disjoint families R_β $(0<\beta<\omega_1)$ of disjoint intervals $x, y, \cdots$ of s, with the following properties. Define $R=\mathbf{U}_\beta R_\beta$, partially order R by $\supset$, and write $\phi(x)=\beta$ to mean $x\in R_\beta$. Then any two elements of R are either disjoint or comparable, $x\supset y$ implies $\phi(x)\leqq\phi(y)$, for every $\beta<\phi(x)$, there is a $y\in R_\beta$ with $y\supset x$, and for every $\beta>\phi(x)$, there are at least two elements $y\in R_\beta$ with $y\subset x$.

The set T of all pairs (x, n) ($x\in R$, n a positive integer) will become the normal Hausdorff space that is not countably paracompact; its topology may be described as follows. First, for every limit ordinal $\beta<\omega_1$, and every $x\in R_\beta$, select a sequence $(f_i(x))$ in R_β such that $f_i(x_1)=f_j(x_2)$ only if $i=j$ and $x_1=x_2$, and such that if $y\supset x$, with $y\neq x$, then y contains almost all $f_i(x)$. Now for every $p=(x, n)$, define $F_i(p)=\{(f_j(x), n-1): j>i\}$ $(=\emptyset$ if $n=1$ or $\phi(x)$ is not a limit ordinal); and for $\alpha<\phi(x)$, define $G_\alpha(p)=\{(y, n): y\supset x,\ \alpha<\phi(y)<\phi(x)\}$ or $\emptyset$ according as $\phi(x)$ is or is not a limit ordinal. Then, inductively, a neighborhood of p is any set of the form

$$\{p\}\cup[\text{a neighborhood of } F_i(p)]\cup[\text{a neighborhood of } G_\alpha(p)].$$

The key in the proof that T has the stated properties is the concept of a Q-full element of R $(Q\subset R)$: $x\in R$ is called Q-full if every successor of x has a successor in Q. Each set

$$L(x, \beta, Q)=\{\text{maximal } y\in Q: y\subset x,\ \phi(y)\geqq\beta\}$$

is countable, so there is a $\delta(x, \beta, Q)<\omega_1$ such that $\phi(y)<\delta(x, \beta, Q)$ for all $y\in L(x, \beta, Q)$.

According to a theorem of Dowker (loc. cit.), a normal space is countably paracompact if and only if (3) if (C_n) is a decreasing sequence of closed sets with empty inter-

section, then there is a sequence of open sets (D_n) with empty intersection, such that each $D_n \supset C_n$. Define $C_n=\{(x, m): m \geq n\}$, and consider any open $D_n \supset C_n$. Define $P_n=\{x: y \subset x \text{ implies } (y, 1) \in D_n\}$; then for all n, every element of R is P_n-full. Choose $x \in R_1$, $\gamma<\omega_1$ such that $\gamma>$every $\delta(x, 1, P_n)$, and $y \in R_\gamma$ such that $y \subset x$; then $(y, 1) \in \bigcap_n D_n$, whence (3) fails.

To establish normality of T (which is clearly a T_1-space), let H, K be any two disjoint closed subsets of T, and define $H_n[K_n]=\{x: (x, n) \in H[K]\}$. Then for all i, j, if x, y are comparable, then either x is not H_i-full or y is not K_j-full. Write $H_0=\bigcup_n H_n$, $K_0=\bigcup_n K_n$ (reviewer's notation). Let $P_1[P_2]=\{x: y \subset x$ implies $y \notin H_0[K_0]\}$, let P_i' be the set of all maximal elements of P_i, and choose $\mu<\omega_1$ so that $\phi(y)<\mu$ for all $y \in P_1' \cup P_2'$. Then if y, z are any two successors of the same $x \in R_\mu$, either $y \notin H_0$ or $z \notin K_0$. It is now easy to separate $H-X$ from $K-X$, where $X=\{(x, n): \phi(x) \leq \mu\}$. An inductive construction, based on the fact that X is countable, supplies a separation of $H \cap X$ from $K \cap X$ [errata, p. 547: line 2, B_m should be B_{m-1}; line 6, first A should be B]. The combination yields disjoint open sets that separate H from K.

Reviewer's comments. The notation is ill-chosen; e.g.: H_n denotes a subset of R, while H denotes a subset of T. There are several clashes between what is written and what is meant; e.g.: the reviewer is still looking for the definition of R_0; "since the segments $\cdots$ are disjoint, they are countable" is false (obviously, every segment is uncountable); "let α be the limit ordinal of $\alpha_1, \alpha_2, \alpha_3, \cdots$" is very confusing (none of the α_n need be a limit ordinal—whereas "let $\alpha=\lim_n \alpha_n$" is clear, short, standard, and expresses the author's intent). There are a great many awkward phrases and circumlocutions, most of which could be shortened and clarified by the use of appropriate mathematical notation; e.g.: the symbol $\subset$ is not used once in the entire paper. All this makes the paper very hard to read. *L. Gillman* (Lafayette, Ind.).

Ball, B. J. A note on the separability of an ordered space. Canad. J. Math. **7** (1955), 548–551.

Let S be a connected linearly ordered space (interval topology); it is shown that each of the following three sets of conditions implies that S is separable: (1, 2) there exists a countable family F of continuous functions on S into S such that each $p \in S$ is an accumulation point of $\{f(p) \mid f \in F\}$, and either (1) S satisfies Souslin's condition (see preceding review), or (2) no element of F has a fixed point; (3) there exists a sequence (f_n) of continuous functions on S into S such that for each $p \in S$, $f_n(p) \to p$, and for infinitely many n, $f_n(p) \neq p$. The proof of (1) is based on a result of Kurepa [C. R. Acad. Sci. Paris **231** (1950), 1113–1114; MR **12**, 397]; (2) and (3) are derived from (1). [Reviewer's remark: a somewhat similar result was obtained by A. G. Walker [J. London Math. Soc. **21** (1946), 9–10; MR **8**, 255, 708]; he used a very strong monotone condition in place of continuity.] *L. Gillman.*

Balachandran, V. K. A mapping theorem for metric spaces. Duke Math. J. **22** (1955), 461–464.

Let f be a (continuous) mapping of a metric space R onto a topological space R^*, and suppose f is both open and closed. The author shows that R^* is then also metrizable, a suitable metric being: $\varrho(x, y)=$Hausdorff distance between the sets $f^{-1}(x)$, $f^{-1}(y)$. [The proof could be simplified by noting that each of the sets $f^{-1}(x)$ is now either open or compact; cf. A. D. Wallace, Amer. J. Math. **61** (1939), 757–763; MR **1**, 30.] He asks whether R^* is metrizable under the weaker hypothesis that f is closed and R^* is first countable (here called "locally separable", a term which may cause some confusion). [The answer is affirmative.] *A. H. Stone* (Manchester).

Strebe, David D. Irreducible closed connexes. Duke Math. J. **22** (1955), 365–372.

Suppose that B is a subset of a connected Hausdorff space H. If no (closed) connected proper subset of H contains B but each proper subset of B is contained in at least one (closed) connected proper subset of H, then B is called a basic set (fundamental set) of H. [Do not confuse with Ayres' use of the term basic set of a continuous curve.] These notions, originally introduced by Gehman and studied in compact (or locally compact) metric spaces, are studied here in more generality. For example, if H has a finite fundamental set of n points and x is a separating point of H, then $H-x$ is the sum of at most n components and each of them intersects F. If H has a fundamental set F which contains all of the non-separating points of H, then F is the unique fundamental set of H. And if H is regular and locally connected, H can have at most one fundamental set. Several more theorems along these lines are proved, but general existence theorems for basic and fundamental sets are (understandably) absent. *F. B. Jones* (Chapel Hill, N.C.).

Ginsburg, Seymour. On a class of pathological functions. Proc. Amer. Math. Soc. **6** (1955), 797–805.

All spaces considered are separable metric. The author studies the problem of approximating a function by dishomeomorphisms; by a dishomeomorphism is meant a function g such that for no set S of power c is g a homeomorphism of S to $g(S)$. The following are the two main results. Theorem 1: For every one-one mapping f of D into E, and every $\epsilon>0$, there is a dishomeomorphism g of D onto $f(D)$ that ε-approximates f (i.e., $d(f(x), g(x))<\varepsilon$ for all x in D). Theorem 4: Suppose that every point of D is a c-condensation point. Let $n>1$, and let f be continuous from D into D such that for all x, $d(x, f^n(x))<\varepsilon/2$, and $|f^{-1}(x)|<c$. Then there is a dishomeomorphism g that ε-approximates f, and that is strongly of period n (i.e., for all x, and all $0<k \leq n$, $g^k(x)=x$ if and only if $k=n$). [That g is a dishomeomorphism is inadvertently omitted from the statement of the theorem.] The following is left open: given $|D|=|E|=c$, does there exist a one-one mapping f of D into E such that for no set A of power c is A homeomorphic to $f(A)$ (not necessarily under f)?
L. Gillman (Lafayette, Ind.).

Anderson, R. D., and Hamstrom, Mary-Elizabeth. On spaces filled up by continuous collections of atriodic continuous curves. Proc. Amer. Math. Soc. **6** (1955), 766–769.

Suppose that G is a nondegenerate continuous collection of atriodic continuous curves (i.e., arcs and simple closed curves) filling up a compact metric continuum M. No totally disconnected closed subset of M separates M. (If each element of G is an arc, the same is true of any connected open subset of M.) Consequently, at no point of M is the dimension of M less than 2. Hence, if G is a one-dimensional continuum, then M is two-dimensional at each of its points. On the other hand, if G is a two-dimensional Cantor manifold, then M is not separated by any rational curve. The "methods do not appear to generalize to higher dimensions." *F. B. Jones.*

de Groot, J. Continuous mappings of a certain family. Fund. Math. **42** (1955), 203–206.

It is proved that there exists a family of 2^c (c the power of the reals) subsets of the plane such that no set of the family can be mapped continuously onto a non-degenerate subset of another set of the family. Kuratowski has proved a corresponding theorem [Topologie, v. I, 3ème ed., Warszawa, 1952, pp. 330–341; MR **14**, 1000] for continuous maps into with c image points; the proof of the present theorem is based upon the proof of Kuratowski.

E. E. Floyd (Charlottesville, Va.).

Novosad, Robert S. Simply connected spaces. Trans. Amer. Math. Soc. **79** (1955), 216–228.

Noting that the monodromy principle does not hold for all simply connected spaces if simple connectivity means the vanishing of the fundamental group, the author adheres to the precept that a space is simply connected if and only if it has no covering space other than itself. Accordingly the possibility of giving a satisfactory definition of simple connectivity depends on the possibility of finding a satisfactory definition of covering space.

The author calls a map p: $(Y, y_0)\to(X, x_0)$ a "generalized covering space" (g.c.s.) if X and Y are arcwise connected, and every map α: $(A, a_0)\to(X, x_0)$, for which a_0 is a deformation retract of A, can be lifted to a unique map $\tilde{\alpha}$: $(A, \alpha_0)\to(Y, y_0)$. The concept does not depend on the choice of points $x_0, y_0 \in p^{-1}(x_0)$. A map of a generalized simply connected space can be lifted to any g.c.s. of the image. There is always a maximal g.c.s. The generalized simply connected spaces satisfy a suitably worded monodromy principle. For connected, locally connected, locally simply connected spaces, generalized simple connectivity coincides with ordinary simple connectivity.

The author states that his purpose is to remove the restriction of X to the class of locally connected spaces implied by the usual definition of covering space [as in, e.g., Chevalley, Theory of Lie groups, v. I, Princeton, 1946; MR **7**, 412]. Actually, the restriction of X to the class of locally connected spaces is only due to an unfortunate wording (while the g.c.s. definition restricts X to the class of arcwise connected spaces). If one defines Y to be a covering space of X if and only if each point x of X has an open neighborhood U such that $p^{-1}(U)$ is the union of a family of pairwise disjoint open sets V each of which is mapped by p topologically upon U [as in, e.g., Ehresmann, Bull. Soc. Math. France **72** (1944), 27–54; MR **7**, 138] one has a perfect definition which seems to have all desired properties (except that one has to give up the guaranteed universal covering). As a matter of fact the author considers this kind of a c.s. (calling it "even covering") and shows that the arc-components of a c.s. are g.c. spaces of their p-images.

A most unfortunate feature of the g.c.s. is that it is not always a local homeomorphism, and in fact the inverse image of a point is not always discrete. (This is apparently the price of a guaranteed universal covering.) A good comparison between g.c.s. and c.s. is afforded by the plane set $4\varrho = \sin(4\pi^2/\theta)$, $0<\theta\leq 2\pi$ (polar coordinates). It has a universal c.s., the graph of $f(x)=\sin(\pi^2/x)$, $0<x\leq 2\pi$; $f(x+2\pi)=f(x)$ (cartesian coordinates), but its universal g.c.s. is the subset of this that lies in the strip $0<x\leq 2\pi$.

The author also attempts to redefine the fundamental group to be the group of covering transformations of the universal covering space (in generalized sense).

R. H. Fox (Princeton, N.J.).

Ganea, Tudor. Transformations continues des espaces euclidiens. Com. Acad. R. P. Române **2** (1952), 413–414. (Romanian. Russian and French summaries)

If φ is a continuous mapping of euclidean n-dimensional space R^n into itself and if the diameters of the sets $\varphi^{-1}(p)$, $p \in R^n$, are bounded, then $\varphi(R^n)$ has a maximal covering space $\tilde{E}$, and its group of covering transformations is finite. *R. H. Fox* (Princeton, N.J.).

Ganea, Tudor. Covering spaces. Acad. R. P. Române. Stud. Cerc. Mat. **1** (1950), 418–471 (1951). (Romanian. Russian and English summaries)

Expository paper on covering spaces and covering transformations, carefully avoiding use of the fundamental group. Most of the results are due to the author, and have appeared elsewhere. *R. H. Fox.*

Ganea, Tudor. Sur la catégorie uni-dimensionelle. Acad. R. P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5** (1953), 127–133. (Romanian. Russian and French summaries)

If E is a locally connected continuum that has a maximal covering $f:\tilde{E}\to E$, the author defines $\mathrm{cat}_1 E$ to be the least integer k such that E is the union of k regions each evenly covered by f. The main theorem is the following: If φ induces an isomorphism of the group of covering transformations of $\tilde{X}$ into the group of covering transformations of $\tilde{Y}$ then $\mathrm{cat}_1 X\leq \mathrm{cat}_1 Y$. Corollaries: (1) if, for every ε, X has an ε-image Y_ε for which $\mathrm{cat}_1 Y_\varepsilon\leq n$, then $\mathrm{cat}_1 X\leq\varepsilon$; (2) if X and Y are quasihomeomorphic, then $\mathrm{cat}_1 X=\mathrm{cat}_1 Y$; (3) if A is a retract of E then $\mathrm{cat}_1 A\leq \mathrm{cat}_1 E$, and equality holds if $f^{-1}(A)$ is connected; (4) if there is a local homeomorphism of X on Y then $\mathrm{cat}_1 X\leq \mathrm{cat}_1 Y$; (5) if $X_1\supset X_2\supset\cdots$, then $\mathrm{cat}_1 \bigcap X_n\leq \liminf \mathrm{cat}_1 X_n$. Finally it shown by an example that if, in the main theorem, one requires the isomorphism induced by φ to be onto, the equality of $\mathrm{cat}_1 X$ and $\mathrm{cat}_1 Y$ need not follow. *R. H. Fox* (Princeton, N.J.).

Ganea, Tudor. Produits symétriques d'espaces topologiques. Com. Acad. R. P. Romîne **4** (1954), 561–563. (Romanian. Russian and French summaries)

Borsuk and Ulam [Bull. Amer. Math. Soc. **37** (1931), 875–882] have raised the question of the dimension of the nth symmetry potency $E(n)$ of a space E. The author appears to have shown that it is the same as the dimension of the Cartesian product E^n, although some misprints have made this part of the paper unreadable. It is shown that $E(n)$ inherits from E certain properties, notably the property of being connected, locally connected and unicoherent, or of being locally contractible. *R. H. Fox.*

Weier, Josef. Die Randsingularitäten von Abbildungen offener Mengen in sich. Math. Ann. **130** (1955), 196–201.

U designates a connected bounded open subset of R^n, $n>1$, such that $\overline{U}$ is "retractable" in the following sense: there is a continuous function $F:\overline{U}\times I\to\overline{U}$ such that $F|\overline{U}\times\{0\}$ is the identity of $\overline{U}$ and $F(U\times(0, 1])\subset U$. A point $a \in \overline{U}-U$ is a "boundary singularity" of $f:U\to U$ if there is a sequence of points $a_i \in U$ converging to a such that $\lim f(a_i)=a$. The author obtains several relationships between fixed points of functions on U and boundary singularities, including the following. If $f:U\to U$ then there is $g:U\to U$, homotopic to f, with at most a finite number of fixed points and no boundary singularities.

If $f:U\to U$ then there is $g:U\to U$, homotopic to f, with no fixed points and at most one boundary singularity. Criteria for "retractability" are not discussed.

S. Stein (Davis, Calif.).

Židkov, N. P. Einige Eigenschaften der diskreten dynamischen Systeme. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 31–59. (Russian)

Sei K ein kompakter metrischer Raum und $\{T_t\}_{-\infty<t<\infty}$ eine einparametrige Gruppe von Homoömorphismen von K auf sich selbst, so daß (K, T_t) ein dynamisches System ist. Ein Unterraum S von K heißt ein Schnitt im Ganzen des Systems, wenn: 1) S einen nichtleeren Durchschnitt mit allen Trajektorien von (K, T_t) hat; 2) $(T_tS)\cap S=0$ für $t\neq kt_0$ $(k=0, \pm1, \pm2, \cdots)$; 3) $Tkt_0S=S$ $(k=0, \pm1, \pm2, \cdots)$. Zuerst wird bewiesen, daß es für einen willkürlichen Unterraum K des n-dimensionalen Einheitswürfels mit einem Homoömorphismus U, $UK=K$, ein stetiges dynamisches System (S, T_t) gibt, welches durch Differentialgleichungen mit stetigen rechten Seiten definiert ist, K als einen Schnitt im Ganzen enthält und in K den Homoömorphismus T induziert. Wenn K kompakt ist, dann kann man etwas mehr aussagen. Der Verf. betrachtet dann minimale abgeschlossene invariante Untermengen eines Systems (S, T_t) und konstruiert ein Beispiel einer solchen Menge R, die nicht homogen ist (d.h. es gibt ein Paar Punkte $p, q\in R$, für welche kein Homoömorphismus U von R auf R existiert, so daß $Up=q$ und $UT_t=T_tU$). Diese Konstruktion löst ein bekanntes Problem von G. D. Birkhoff. In der Konstruktion werden die symbolischen Trajektorien von Morse und Hedlund [Amer. J. Math. **60** (1938), 815–866] benutzt. Verf. führt auch den Begriff der Zerlegbarkeit einer minimalen Menge ein. Wenn K minimal in bezug auf T ist, aber nicht minimal in bezug auf eine gewisse Potenz T^n, dann heißt K zerlegbar. Einige Sätze über solche Mengen werden bewiesen. Schließlich diskutiert Verf. einige Spektraleigenschaften der minimalen Mengen.

E. Hewitt (Zbl **52**, 341).

Markus, L. Line element fields and Lorentz structures on differentiable manifolds. Ann. of Math. (2) **62** (1955), 411–417.

The author considers continuous fields of line elements, with possible isolated singularities, on a differentiable manifold M of dimension n. A field L may be called orientable if it is determined by a vector field, and the author gives criteria for orientability in terms of the fundamental group (i) of M for $n>2$ and (ii) of M minus the points of singularity for $n=2$. He then relates the existence of a non-singular field on M to several other phenomena such as the existence of a globally defined hyperbolic differential equation. Finally in the case of a field with singularities at $P_1, \cdots, P_k$ on a compact M he defines an index $i(P)$ at each singularity and proves $\sum_{j=1}^{k} i(P_j)=2\chi(M)-N$, where $\chi(M)$ is the Euler characteristic and N $(=0$ for $n>2)$ is the number of non-orientable singularities. *W. M. Boothby* (Evanston, Ill.).

Harary, Frank. On local balance and N-balance in signed graphs. Michigan Math. J. **3** (1955), 37–41.

A signed graph is obtained from a linear graph when some of its lines are regarded as positive and the remaining lines as negative. The sign of a cycle is the product of the sign of its lines. A signed graph is called balanced at the point P if all cycles containing P are positive, and it is called N-balanced if all cycles of length $\leq N$ in the graph are positive. Relations between these two kinds of balance and the structure of the graph are derived, for example Theorem 1 shows the relation between balance at a point and articulation points. *G. A. Dirac* (Vienna).

Ore, Oystein. Graphs and matching theorems. Duke Math. J. **22** (1955), 625–639.

Let V and V' be disjoint sets of vertices of a locally finite even graph G such that each edge of G has an end in each of them. If A is a finite subset of V let $V'(A)$ be the set of all vertices of V' joined to A. Define $V(A')$ analogously for $A'\in V'$. The author makes a thorough investigation of the relation between V and V' in terms of the pairs $\{A, V'(A)\}$ and $\{A', V(A')\}$. He pays special attention to the deficiency of a set A, that is the excess of the number of members of A over the number of members of $V'(A)$.

The author's main result is that if δ and δ' are non-negative integers, being the greatest deficiencies in V and V' respectively, then all but δ of the members of V can be paired off with all but δ' of the members of V' so that the members of each pair are joined in G. This generalizes the results of P. Hall [J. London Math. Soc. **10** (1935), 26–30] and R. Rado [Quart. J. Math. Oxford Ser. **20** (1949), 95–104; MR **10**, 728] on the 1-factors of even graphs. *W. T. Tutte* (Toronto, Ont.).

Algebraic Topology

Schirmer, Helga. Mindestzahlen von Koinzidenzpunkten. J. Reine Angew. Math. **194** (1955), 21–39.

The author considers pairs of mappings $\mathfrak{M}^n\to\mathfrak{M}'^n$, where $\mathfrak{M}$ and $\mathfrak{M}'$ are triangulated orientable manifolds. (f_1, f_2) is homotopic to (g_1, g_2) if f_i is homotopic to g_i, $i=1, 2$. A point p of $\mathfrak{M}^n$ is a coincidence point for (f_1, f_2) if $f_1(p)=f_2(p)$. To each set of coincidence points there is attached an index in a well-known manner. The coincidence points of (f_1, f_2) fall into coincidence classes, p and q belonging to the same class if there is a path $\alpha=pq$ such that $f_1(\alpha)(f_2(\alpha))^{-1}$ is null-homotopic. The main theorem asserts that each homotopy class of mapping-pairs contains a pair (f_1, f_2) for which each coincidence class of index $\neq0$ contains just one point and there are no coincidence points of index zero. *P. A. Smith.*

Aoki, Kiyoshi. Note on deformation retract. Proc. Japan Acad. **30** (1954), 538–541.

This paper may be regarded as a collection of lemmas to be used in a later paper [same Proc. **30** (1954), 694–697; MR **16**, 948]. *R. H. Fox* (Princeton, N.J.).

Chen, Kuo-Tsai. Iterated integrals and exponential homomorphisms. Proc. London Math. Soc. (3) **4** (1954), 502–512.

In m-dimensional Euclidean space E the paths that have the origin as initial point can be made into a group $\mathfrak{P}$ as follows: two such paths α, β are multiplied by translating β so that its initial point coincides with the end-point of α; then $\alpha\beta$ is the path obtained by traversing α and then the translated β. The inverse of a path α is obtained by translating the end-point to the origin and traversing the translated α backward. To make a group it is necessary to be unable to distinguish between the empty path 1 and a path of the form $\alpha\cdot\alpha^{-1}$. Given a path $\alpha: t\to(x_1(t), \cdots, x_m(t))$, $a\leq t\leq b$, an iterated integral is

defined:

$$I_{i_1\cdots i_p}(\alpha)=\int_a^b\int_a^{t_p}\cdots\int_a^{t_2}dx_{i_1}(t_1)\cdots dx_{i_{p-1}}(t_{p-1})dx_{i_p}(t_p).$$

A multiplicative homomorphism θ, called the exponential homomorphism, of $\mathfrak{P}$ into the ring Λ of formal power series over the reals R, in the noncommuting variables $X_1, \cdots, X_m$, is defined by $\theta(\alpha)=1+\sum_{p\geq 1} I_{i_1\cdots i_p}(\alpha)\cdot X_{i_1}\cdots X_{i_p}$. The normal subgroup of $\mathfrak{P}$, consisting of those α for which $\theta(\alpha)=1+$terms of order p or more, is denoted by $\mathfrak{P}_p$, and it is shown and $\mathfrak{P}_p/\mathfrak{P}_{p+1}$ is isomorphic with the additive group of a vector space of finite dimension over R. If α is the straight path from the origin to $(C_1, \cdots, C_m)$ then $\theta(\alpha)=\sum_{p=0}^{\infty}(\sum c_i X_i)^p/p!=\exp(\sum c_i X_i)$.

Let $\mathfrak{F}$ be the free product of the multiplication group of positive reals ($\approx$ the additive group of all reals) R with itself m times. To each word $u=e_{i_1}^{p_1}\cdots e_{i_q}^{p_q}$ of $\mathfrak{F}$ associate a rectangular path $\sigma(u)$ describing successively μ_j units in the direction of the x_{ij}-axis, $j=1, \cdots, q$. Then σ is an isomorphism and $\Theta=\theta\sigma$ maps $\mathfrak{F}$ into Λ with $\theta\sigma(e_i^{\mu})=\exp \mu X_i$.

In general, "exponential homomorphisms" are defined from certain free products to certain formal-power-series rings. A specialization of this generalization leads to the Magnus series, in which the normal subgroups $\mathfrak{P}_p$ become the terms of the lower central series. *R. H. Fox.*

Toda, Hirosi. Quelques tables des groupes d'homotopie des groupes de Lie. C. R. Acad. Sci. Paris **241** (1955), 922–923.

The rth homotopy groups, $9\leq r\leq 15$, of the following simple Lie groups are announced in this note: $SO(n)$, $12\leq n\leq 17$; $SU(n)$, $3\leq n\leq 8$; $Sp(n)$, $2\leq n\leq 4$; $Spin(7)$, $Spin(9)$; G_2 and F_4. A few lines of brief indications of the proofs are also given in this note. *S. T. Hu.*

Wada, Hidekazu. On the asphericity of the higher dimensional complex. Tôhoku Math. J. (2) **7** (1955), 9–13.

The author considers the question: when can a non-aspherical n-dimensional polytope be made aspherical by attaching n-cells? Cockcroft [Proc. London Math. Soc. (3) **4** (1954), 375–384; MR **16**, 62] has solved this in case $n=2$. The author gives a solution for $n\geq 3$: Let K be a polytope obtained by attaching n-cells to an n-dimensional polytope L. K is aspherical if and only if the following three conditions are satisfied: (1): $\pi_r(L)=0$ for $1<r\leq n-2$ (if $n\geq 4$), (2): $\pi_{n-1}(L)$ is a non-zero free $\pi_1(L)$-module, and (3): $H_n(\tilde{L})=0$, $\tilde{L}$=the universal covering space of L. It is further shown that, if K is aspherical, then $\tilde{L}$ must belong to the homotopy type of the union of $(n-1)$-spheres with a single point in common; the converse is not, however, true. *J. Dugundji* (Los Angeles, Calif.).

Zaĭdenman, I. A. Über die Fundamentalgruppe der Summe zweier zusammenhängender Polyeder mit nicht zusammenhängendem Durchschnitt. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 69–71. (Russian)

Verf. beweist die folgende Verallgemeinerung eines bekannten Satzes: Sei das zusammenhängende Polyeder K^* die Summe zweier zusammenhängender Polyeder K und K', deren Durchschnitt aus den k Komponenten $A_1, \cdots, A_k$ besteht. Seien f_i: $F(A_i)\to F(K)$ $(i=1, \cdots, k)$ Homomorphismen der Fundamentalgruppe $F(A_i)$ in die Fundamentalgruppe $F(K)$, die durch Verbindung des Anfangspunktes von $F(K)$ mit dem von $F(A_i)$ durch einen Weg in K hergestellt werden. Entsprechend seien f_i': $F(A_i)\to F(K')$ $(i=1, \cdots, k)$ definiert. Dann ist die Fundamentalgruppe $F(K^*)$ die Faktorgruppe des freien Produktes $F(K)\circ F(K')\circ C^{(k-1)}$ ($C^{(k-1)}$=freie Gruppe mit $k-1$ Erzeugenden $c_1, \cdots, c_{k-1}$) nach dem Normalteiler H, der von den Elementen $f_i(a_i)c_i^{-1}(f_i'(a_i))^{-1}\cdot c_i$ und $f_k(a_k)(f_k'(a_k))^{-1}$ mit $(a_i \in F(A_i))$ erzeugt wird.

E. Burger (Zbl **52**, 397).

Barratt, M. G. Homotopy ringoids and homotopy groups. Quart. J. Math., Oxford Ser. (2) **5** (1954), 271–290.

The author defines the notion of ringoid as follows: Let I be an index set, and suppose that for $i, j \in I$ there is given an abelian group $G_{i,j}$. A ringoid is $\{G_{i,j}\}$ together with pairings $G_{i,j}\times G_{j,k}\to G_{i,k}$ such that each $G_{i,i}$ has a unit element. Let r, N be integers such that $r>1$, and $N\leq 2r-2$. Denote by C_r^N a collection of $(r-1)$-connected CW-complexes of dimension not exceeding N, and by (C_r^N) the collection of homotopy classes of maps between the spaces of C^N. For $X_i, X_j \in C_r^N$ let $G_{i,j}$ be the set of homotopy classes of maps $X_j\to X_i$. The author first turns each $G_{i,j}$ into a group by observing that the homotopy classes of maps $X_j\to X_i$ correspond in a natural 1:1 fashion with the homotopy classes of maps $SX_j\to SX_i$, where SX_i denotes the suspension of X_i, or even to the homotopy classes of maps $SSX_j\to SSX_i$. In the last-mentioned case the homotopy classes of maps are known to form an abelian group under track addition [M. G. Barratt, Proc. London Math. Soc. (3) **5** (1955), 71–106; MR **17**, 290; E. Spanier and J. H. C. Whitehead, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660; MR **15**, 52]. He now turns $G_{i,j}$ into a ringoid by using composition of maps to give a pairing $G_{i,j}\times G_{j,k}\to G_{i,k}$.

Let Y_p^{n+1} be the space obtained by attaching an $(n+1)$-cell to an n-sphere S^n by a map of degree p. The author defines maps $\varphi_{pq,p}: Y^{n+1}\to Y_{pq}^{n+1}$ which map the $(n+1)$-cell homeomorphically, and the n-sphere by a map of degree q, maps $\psi_{p,pq}: Y_{pq}^{n+1}\to Y_p^{n+1}$ which map the $(n+1)$-cell with a map of degree q and the n-sphere homeomorphically, and maps $\mu_{q,p}: Y_p^{n+1}\to Y_q^{n+1}$ where $\mu_{q,p}=i_q\circ\alpha'\circ\mu'$, $i_q: S^n\to Y_q^{n+1}$ is the inclusion map, $\alpha': S^{n+1}\to S^n$ is essential, and $\mu': Y_p^{n+1}\to S^{n+1}$ collapses the n-sphere in Y_p^{n+1} to a point.

The main theorem of the author now says that if Y^{n+1} is the set of Y_q^{n+1}, $n\geq 3$, then the ringoid (Y^{n+1}) is generated by i_p, $\mu_{p,p}$, $\varphi_{pq,p}$, $\psi_{p,pq}$ for all $p>0$, $q>1$. Further, the relations between generators are given. The remainder of the paper is chiefly devoted to obtaining relations between the homotopy groups $\pi_m(Y_q^{n+1})$, $\pi_m(S^n)$, and $\pi_{m-1}(S^n)$. This is partly done by studying the ringoid (Y^{n+1}, S^m) obtained by adjoining the m-sphere to the set Y^{n+1}. *J. C. Moore* (Princeton, N.J.).

Barratt, M. G. Track groups. II. Proc. London Math. Soc. (3) **5** 1955), 285–329.

Let K be a finite CW complex with 1-vertex, and K^q its q-skeleton. Further, let X be a topological space, and x_0 a point of X. Denote by $(p, q)^m$ the m-dimensional homotopy group of the function space of maps $f: (K^p, K^q)\to (X, x_0)$, the base point for the homotopy group being the constant map [Barratt, same Proc. Math. Soc. (3) **5** (1955), 71–106; MR **17**, 290]. Denote $H^n(K^{n+1}, K^{n-1}, G)$ (the n-dimensional cohomology group of K^{n+1} modulo K^{n-1} with coefficients in G) by $H^n(G)$. The author proved in part I that there was for $m+n>2$, an exact sequence of abelian groups

$$0\to H^{n+1}(\pi_{m+n+1})\to(n+1, n-1)^m\to H^n(\pi_{m+n})\to 0.$$

Now $H^n(\pi_{m+n}) \cong H^n(Z) \otimes \pi_{m+n} + \mathrm{Tor}(H^{n+1}(Z), \pi_{m+n})$. Using such a decomposition of $H^n(\pi_{m+n})$, let G_1 be the counter image in $(n+1, n-1)^m$ of $H^n(Z) \otimes \pi_{m+n}$, and G_2 the counter image of $\mathrm{Tor}(H^{n+1}(Z), \pi_{m+n})$. In this paper the author proves $G_1 = H^{n+1}(\pi_{m+n+1}) + H^n(Z) \otimes \pi_{m+n}$, and determines the group extension necessary to describe G_2 in terms of a set of generators of $\mathrm{Tor}(H^{n+1}(Z), \pi_{m+n})$.

By utilizing the preceding general result the author calculates the group $(n+1, n-1)^m$ explicitly in case K is an n-sphere with an $(n+1)$-cell attached, and X is either an $(n+m)$-sphere with an $(n+m+1)$-cell attached, or a k-sphere where $k \geqq n+m$.

The paper concludes with a section on normal spaces wherein coverings and inverse limits of nerves of coverings are studied in order to extend the domain of validity of the results of this and his preceding paper on track groups.

J. C. Moore (Princeton, N.J.).

Barratt, M. G., and Whitehead, J. H. C. On the exact couple of a CW-triad. Proc. London Math. Soc. (3) **5** (1955), 330–341.

It was shown by J. H. C. Whitehead [Ann. of Math. (2) **52** (1950), 51–110; MR **12**, 43] that if X is a CW complex, then there is an exact sequence

$$\cdots \to \pi_q(X) \to H_q(X) \to \Gamma_{q-1}(X) \to \pi_{q-1}(X) \to \cdots,$$

where $\Gamma_q(X)$ is the image of i: $\pi_q(X^{q-1}) \to \pi_q(X^q)$, the homomorphism induced by the inclusion map of the $q-1$ skeleton X^{q-1} into the q skeleton X^q of X.

The main purpose of this paper is to introduce an exact couple [W. S. Massey, ibid. **56** (1952), 363–396; MR **14**, 672] associated with a CW triad $(X; A, B)$, and by using this exact couple define an exact sequence

$$\cdots \to H_{m+1}(A, C; H_n(B, C)) \to \Gamma_{m+n-1}(X; A, B) \to$$
$$\to \pi_{m+n-1}(X; A, B) \to H_m(A, C; H_n(B, C)) \to \cdots,$$

where $C = A \cap B$, and $\Gamma_{m+n-1}(X; A, B)$ is the appropriate generalization of $\Gamma_q(X)$. In the course of this procedure the authors: 1) define a natural homomorphism μ: $\pi_{m+n-1}(X; A, B) \to H_m(A, C; H_n(B, C))$; 2) develop some additional properties of triad homotopy groups; and 3) prove that the derived exact couple of the exact couple of a CW triad is a homotopy invariant. *J. C. Moore.*

James, I. M. Reduced product spaces. Ann. of Math. (2) **62** (1955), 170–197.

Let A be a countable CW-complex with one vertex a_0. Denote by $\hat{A}$ the suspension of A, and by Ω the space of loops in $\hat{A}$. Recall that A is naturally imbedded in Ω. The main purpose of this paper is to define the reduced product complex A_∞ of A, and derive some of its properties. Points of A_∞ are finite sequences of points of $A - a_0$, and it is a countable CW-complex with one vertex whose cells are finite sequences of cells of $A - a_0$. The original complex A is a subcomplex of A_∞, and there is a map $f : (A_\infty, A) \to (\Omega, A)$ which is the identity on A, and such that $f^\# : \pi_q(A_\infty, A) \to \pi_q(\Omega, A)$ for all q. Further there is a multiplication in the complex A_∞, and the map f induces an isomorphism of the Pontrjagin ring $H_*(A_\infty)$ with the Pontrjagin ring $H_*(\Omega)$.

Almost all questions involving Ω can be considered by looking instead at A_∞, and due to the simple structure of A_∞ they are often easier to solve there. *J. C. Moore.*

Hirsch, Guy. Quelques propriétés des produits de Steenrod. C. R. Acad. Sci. Paris **241** (1955), 923–925.

Let $C^* = \sum C^p$ denote the graded ring of cochains of a simplicial polyhedron K with coefficients in an associative ring R, and let $H^* = \sum H^p$ denote the cohomology ring of K over R. For any cohomology classes $u \in H^p$, $v \in H^q$, $w \in H^r$ such that $uv = vw = 0$, let $\langle u, v, w \rangle$ denote the "triple product" introduced by the reviewer [Bull. Amer. Math. Soc. **57** (1951), 74]; $\langle u, v, w \rangle$ may be considered to be an element of the quotient group

$$H^n / (H^{p+q-1} \cdot w + u \cdot H^{q+r-1}) \text{ where } n = p+q+r-1.$$

The main theorem of this note asserts that if the coefficient ring R is the ring of integers modulo 2, and x and y are cohomology classes such that $xy = 0$, then $y(\mathrm{Sq}_1 x)$ is a representative of the triple product $\langle x, y, x \rangle$. Here Sq_1 denotes the Steenrod square [Ann. of Math. (2) **48** (1947), 290–320; MR **9**, 154]. In order to prove this theorem, the author gives two identities involving Steenrod's cup-i products and ordinary cup products. The first of these identities asserts that for any fixed cochain a, the mapping $x \to x \cup_1 a$ is an anti-derivation of the ring of cochains.

W. S. Massey (Providence, R.I.).

Wu, Wen-tsün. On squares in Grassmann manifolds. J. Chinese Math. Soc. **2** (1953), 205–230. (Chinese. English summary)

This paper gives detailed proofs of results announced previously [C.R. Acad. Sci. Paris **230** (1950), 918–920; MR **12**, 42]. We determine completely the squares in Grassmannian manifolds. As a consequence we prove that for an $(m-1)$-sphere bundle the mod 2 Stiefel-Whitney classes are all determined, with the aid of cup products and squares, by those of dimension a power of 2.

Autoreferat (Zbl **51**, 141).

Šura-Bura, M. R. Projection spectra and spaces defined by them. Mat. Sb. N.S. **36(78)** (1955), 93–110. (Russian)

Continuing the work of P. S. Aleksandrov [Ann. of Math. (2) **30** (1928), 101–187] and A. Kuroš [Compositio Math. **2** (1935), 471–476], the author considers the general projection spectrum $K = \{K_\alpha, \varrho_\alpha^\beta\}$; here α, β run through a directed set A, K_α is a finite simplicial complex, ϱ_α^β is a collection of simplicial maps from K_α to K_β for $\beta > \alpha$, the set of ϱ_α^β is closed under composition, the various ϱ_α^β-images of a simplex x of K_β have a smallest common carrier simplex of K_α, $\pi_\alpha^\beta x$ means the complex consisting of this carrier simplex and its faces. A projective set in the spectrum K is a set such that for any finite subset x_α the π_α^β-inverse images have non-empty intersections for arbitrarily large β. A thread is a maximal projective set; its intersection with any K_α consists of a simplex (called its representative in K_α) and all its faces. The set of threads is made into a compact T_1 space by defining neighborhoods (all threads whose representative at a finite number of indices are faces of the representatives of the given thread). A spectrum is an H-spectrum if for any two threads there is an α such that the two representatives in K_α have no face in common; the corresponding space is then T_2.

A simplex is essential if it is π_α^β-image for arbitrarily large β. The essential subspectrum defines the same space, provided it and K are H-spectra. A cofinal subspectrum defines the same space again provided it and K are H-spectra. Examples show that it is not sufficient to assume the corresponding spaces T_2. For families of finite closed coverings of a (compact T_1) space the concepts "directed", "arbitrarily fine", "full" are defined; they are somewhat more restrictive than the usual definitions. Any family

defines a spectrum in the customary manner (nerves of the coverings), and it is shown that for directed, arbitrarily fine families the space associated with the spectrum is homeomorphic with the original space. Moreover, a space R^* is image of a space R if and only if it is homeomorphic with the space associated with the spectrum of some directed full family for R. A spectrum K defines a family of coverings of its associated space in a natural way; this family is investigated. In particular, K is an H-spectrum if and only if this family is arbitrarily fine.

H. Samelson (Ann Arbor, Mich.).

Aleksandrov, P. Der topologische Dualitätssatz von Pontrjagin. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 3–29. (Russian)

Dies ist ein sehr ausfürlicher und leicht lesbarer Beweis des Dualitätssatzes von Alexander-Pontrjagin. Zunächst werden die benötigten Hilfsmittel aus der Topologie und der Charakterentheorie zusammengestellt. Dann wird der Dualitätssatz für den Fall (krummer) Polyeder im R^n im wesentlichen wie bei Alexandroff und Hopf [Topologie, Bd. 1, Springer, Berlin, 1935] bewiesen, wobei jedoch durch konsequente Verwendung der Charakterentheorie mancherlei Vereinfachungen erzielt werden. Darauf wird die Theorie der Gruppenspektren im weiterhin benötigten Umfang ausführlich dargestellt und schließlich der allgemeine Dualitätssatz sehr einfach aus dem speziellen nach der vom Verf. schon früher [Uspehi Mat. Nauk (N.S.) 6 (1951), no. 5(44), 195–196] skizzierten Methode gewonnen.

E. Burger (Zbl **52**, 397).

James, Robert C. Combinatorial topology of surfaces. Math. Mag. **29** (1955), 1–39.

This paper expounds (1) the topological classification of surfaces and (2) parts of the theory of covering surfaces with special attention to Riemann surfaces.

S. S. Cairns (Urbana, Ill.).

El'sgol'c, L. E. Genaue Abschätzung der Anzahl der geometrisch und analytisch verschiedenen kritischen Punkte. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 61–68. (Russian)

Sei M eine n-dimensionale geschlossene Mannigfaltigkeit und $\tilde{E}1\,M$ die Minimalzahl q abgeschlossener Elemente E_i $(i=1, \cdots, q)$ mit folgenden Eigenschaften: 1) $\sum_1^q E_i = M$; 2) $D_{s+1}=(\sum_1^s E_i)\cap E_{s+1}$ $(s=1, \cdots, q-1)$ ist das topologische Produkt einer p-Sphäre mit einem $(n-p-1)$-Element bzw. im Falle $p=n$ eine $(n-1)$-Sphäre; 3) für $p\neq n$ existiert eine beliebig kleine abgeschlossene Umgebung der Menge D_{s+1}, genommen bezüglich des Randes von $\sum_1^s E_i$, welche monoton isotop ist zu D_{s+1} auf dem Rand von $\sum_1^s E_i$, sowie eine echte Teilmenge von D_{s+1} mit der gleichen Isotopie-Eigenschaft. Verf. beweist, daß $\tilde{E}1\,M$ gleich der unteren Grenze der Anzahlen der nichtausgearteten kritischen Punkte von Funktionen auf M ist. Falls $\tilde{E}1\,M$ gleich der Summe der Bettizahlen mod 2 von M ist, so existiert zu vorgegebenen Anzahlen, die den Morseschen Ungleichungen genügen, eine Funktion auf M, die die vorgegebenen Anzahlen von kritischen Punkten der einzelnen Typen besitzt. Auch die untere Grenze der Anzahlen der geometrisch verschiedenen kritischen Punkte der stetigen Funktionen auf M kann in ähnlicher Weise bestimmt werden. Ferner werden diese Sätze auch übertragen auf den Fall der Abbildungen von M auf die Kreislinie.

E. Burger (Zbl **52**, 335).

See also: Chevalley, p. 342; Frölicher, p. 409.

GEOMETRY

★**[Anspach, Pierre A. L.] Aperçu de la théorie des polygones réguliers.** Bruxelles, 1955. 92 pp.

The book deals with the regular polygons of 7, 11, and 13 sides. Equations are derived giving the lengths of the sides of those polygons. Thus the roots of the equation $x^3-x^2-2x+1=0$ are the lengths of the sides of the three regular heptagons, one of them convex and the other two stellar, that may be inscribed in a circle of radius unity. [The Arabic geometer Aboûl Djoûd obtained this equation, for the same purpose, about the middle of the eleventh century. See Fr. Woepke, L'Algèbre d'Omar Alkhayyâmî, Duprat, Paris 1851 pp. 126, 127.]

As an example of the propositions obtained, the following may be quoted (art. 18). Property of the regular convex heptagon inscribed in the circle of radius unity. We move on the circumference in a continuous manner, and we consider the consecutive vertices S_1, S_2, S_3, S_4 numbered in the order in which they are encountered. The midpoint of the chord S_1S_4 is J, the midpoint of the chord S_2S_3 is K. We make the arc S_2S_3 slide on the circumference, the amplitude of the motion being half the amplitude of the arc. The square of the segment joining the point J (which remained fixed) to the point K in its new position is equal to one half.

To avoid excessive lengths, the author refrains from verbalizing the geometrical situation involved in the propositions, and instead he uses the expedient of pointing at a figure, of which there are fifteen, and stating in words only the final conclusion.

About half of the space available is devoted to the study of a special triangle, which the author names heptal, its sides being the sides of the three regular heptagons which may be inscribed in the same circle. This study is thorough and imaginative. The close connection which this study reveals between the lemniscate of Bernoulli and that triangle, and the heptagon in general, is rather surprising.

The heptal triangle deserves more attention than it has received, but it has not failed to be noticed. Thus Liénard called attention to it by the ratios of its angles 1:2:4 [Mathesis 3 (1883), 231; 5 (1885) 92]. By Gikot [Bull. Sci. Math. Phys. Elém. 9 (1904), 296–297] it was considered as the triangle similar to its orthic triangle. Gikot's solution of this problem concludes with the following note: „Je ne connais pas de triangle plus remarquable que celui-la; c'est une vraie mine d'or ··· Il vaudrait la peine de fouiller ce triangle méthodiquement." The book under review amply fulfills this desideratum.

The method of investigation is analytic and makes use of barycentric coordinates. Proofs in this Aperçu are frequently omitted, others are given outline, none in full detail.

The author is quite emotional about his work, and when he announces his results he waxes almost estatic. To the more remarkable points he likes to give fancy names, like Arg, Lol, Event, ··· (shades of Desargues!).

In the introduction the author states that his complete work, of which the book under review is an Aperçu, consists of nineteen books (that is, large subdivisions). The Aperçu is published (two-letter printing) by the author himself, from whom, presumably, copies may be

secured (94, rue Berckmans, Saint Gilles, Bruxelles, Belgium). *N. A. Court* (Norman, Okla.).

Ordered 6-11-56

★**Salkowski, E. Darstellende Geometrie. 5te Aufl. Bearbeitet von Walter Schulze.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1955. x+213 pp. DM 9.50.

This is a revision of the fourth edition of Salkowski's text book [1949; MR **11**, 126]. The figures were redrawn and improved and some new figures were added to provide more illustrative examples. Even though the number of figures was increased appreciably (from 318 to 367) the textual material was hardly changed. *E. Lukacs.*

Thébault, Victor. Recreational geometry. Geodesics. Scripta Math. **21** (1955), 146–158.

Goormaghtigh, R. L'orthopôle dans les polygones inscriptibles. Mathesis **64** (1955), 261–266.

Court, N. A. Three mutually orthogonal real circles. Amer. Math. Monthly **62** (1955), no. 7, part II, 59–65.

A sphere meets three mutually orthogonal planes through its center in three mutually orthogonal great circles, whose six points of intersection are the vertices of a regular octahedron. Each face of the octahedron is adjacent to three others, alternate to three others, and opposite to the remaining one. Two opposite faces lie in parallel planes and are in perspective from a line at infinity. The circumcircles of two alternate faces touch, those of two adjacent faces are orthogonal, and those of two opposite faces are parallel. Stereographic projection yields a set of three mutually orthogonal circles in a plane, and eight other circles through their six points of intersection. Each of the eight circles touches three others and is orthogonal to three others, while its radical axis with the remaining one is the axis of perspectivity of the corresponding triangles.

The author derives these and many other properties without using stereographic projection.

H. S. M. Coxeter (Toronto, Ont.).

Baur, A. Die Sätze von Pascal und von Brianchon. Math.-Phys. Semesterber. **4** (1955), 281–288.

Wanka, K. Die Konfiguration des Pascalschen Sechseckes. Elem. Math. **10** (1955), 128–130.

Justinijanović, Juraj. Die Lösung zweier metrischen Fundamentalaufgaben in der orthogonalen Axonometrie ohne Umlegungen. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 41–46. (Serbo-Croatian summary)

Marmion, A. Sur un théorème de A. de Majo. Mathesis **64** (1955), 242–249.

Deaux, R. Sur l'antigraphie. Mathesis **64** (1955), 254–261.

Stamatis, Evangelos. Geometrischer Beweis der archimedischen Näherungswerte für $\sqrt{3}$. Prakt. Akad. Athēnōn **30** (1955), 255–262. (Greek. German summary)

Szerszeń, S. Some properties of the foci of a conical, discovered by means of de la Fresnaye's perspectograph. Prace Mat. **1** (1955), 292–322. (Polish. Russian and English summaries)

Droussent, L., et Deaux, R. Sur le faisceau de Griffiths. Mathesis **64** (1955), 225–231.

Ostrowski, A. M. Note on a theorem by A. Brauer. Duke Math. J. **22** (1955), 469–470.

The paper gives a simpler proof of an inclusion property of Cassini curves [Brauer, same J. **19** (1952), 75–91; MR **13**, 813]. *W. Givens* (Knoxville, Tenn.).

Moscovici, Moses. Une interprétation géométrique naturelle des solutions complexes de certains problèmes de géométrie. Com. Acad. R. P. Romîne **5** (1955), 943–947. (Romanian. Russian and French summaries)

Rosenfeld, Azriel. An axiomatic triangular geometry. Amer. Math. Monthly **62** (1955), no. 7, part II, 52–58.

The author gives a system of axioms for the affine plane [cf. Coxeter, Scripta Math. **21** (1955), 5–14; MR **16**, 949] using the signed area of a triangle as a primitive concept.

H. S. M. Coxeter (Toronto, Ont.).

Andreoli, Giulio. Geometrie riducibili non lineari di un simplesso e geometria booleana di un ipercubo. Giorn. Mat. Battaglini (5) **3(83)** (1955), 13–40.

A discussion of geometrical interpretations of finite Boolean algebras via cells of simplices, hypercubes, etc. Particular attention is devoted to the ternary median operation of Kiss, and to effects of groups of symmetries.

G. Birkhoff (Cambridge, Mass.).

Bompiani, E. Omografie e quadriche. Rend. Mat. e Appl. (5) **14** (1955), 359–367.

A collineation $\mathbf{y}=\mathbf{mx}$ of S_n is said to be "adherent" to a quadric $\mathbf{x}^T\mathbf{ax}=\mathbf{0}$ if the join of every point $\mathbf{x}$ of the quadric to its transform $\mathbf{y}$ by the collineation touches the quadric. This requires that

$$\mathbf{am}+\mathbf{m}^T\mathbf{a}=\lambda\mathbf{a},$$

where λ is a scalar (possibly zero), and the same condition expresses also that the given quadric is the coincidence locus (when this exists) of the correlation of matrix $\mathbf{am}$. The author observes that this relation of adherence generalizes an earlier concept of "inherence" due to B. Segre [Ann. Mat. Pura Appl. (4) **12** (1934), 327–347]. Some observations are made about the class of collineations adherent to a given quadric. *J. G. Semple.*

Stubban, John Olav. Sur la géométrie de direction. Norske Vid. Selsk. Skr., Trondheim **1954**, no. 1, 67 pp.

This paper begins with a general formulation, based on linear transformation groups, of the concept of a pair of semi-spaces S_n and $-S_n$ associated with (or based on) a given projective space S_n which is either itself real or the transform of a real space by a complex collineation. The limitation is that n must be odd. This leads by duality to the concept of a pair of opposite semi-points P and $-P$ based on any point P of a suitable $(n+1)$-dimensional space S_{n+1}. The author then defines a Laguerre geometry of semi-primes (oriented primes) in any real non-euclidean space ζ of even dimension m, the definition bring suitably adapted later to the case when ζ is euclidean. The basic construction is the projection of a real quadric primal Ω in a space S_{m+1} from an external point O on to the polar prime ζ of O with respect to Ω; the two points X_1, X_2 in which Ω is met by a line through O project separately, in a way which is defined, into opposite semi-points X and $-X$ in ζ; and these are carried by a fixed corre-

lation into opposite semi-primes $\bar{x}$ and $-\bar{x}$ of ζ, as required. The geometry in ζ is non-euclidean or euclidean according as Ω is a non-singular quadric or a point-cone. The basic object in ζ is the semi-sphere (rather than the point), image of a prime section of Ω, and various properties of these semi-spheres are given.

A Cremona transformation of the space of oriented primes, such as ζ, into another or the same space ζ' of the same kind is defined by any ordinary birational transformation of the quadric Ω into another or the same quadric Ω'. The rest of the paper is very largely taken up by detailed discussions of such birational transformations of Ω, reducing by projection to ordinary Cremona transformations of a space S_m, and by interpretation of the results in terms of the non-euclidean or euclidean spaces of oriented primes ζ, ζ'. The paper contains a considerable number of puzzling misprints and is not easy to follow.

J. G. Semple (London).

Lagrange, René. Les congruences planes de coniques qui n'ont que deux points focaux. Acta Math. **93** (1955), 257–292.

The author's work on congruences of circles [Bull. Sci. Math. (2) **71** (1947), 82–104; MR **9**, 617] applies through projective transformation to congruences of conics with two distinct fixed points, and suggested this more general memoir. In chapter I here the author investigates the possible number of focal points for a congruence of algebraic curves of given degree with simple or multiple fixed points. In particular, the only congruences which have two moving, simple focal points, distinct or not, are congruences of unicursal curves without moving multiple points. Such congruences of conics have two fixed points. Congruences of cubics are treated briefly and they suggest more generally that if a regular point of the curve is a multiple focal point in the congruence, it usually describes a curve instead of a two-dimensional domain.

From here on the author restricts himself to congruences of conics with two focal points. In chapter II he builds up the detailed theory of the focal points. In particular, when both focal points generate two-dimensional domains the correspondence between associated focal elements is an interesting contact transformation. In chapter III the author takes up congruences where one of the focal points determines a curve. The congruence decomposes into a simple infinity of pencils, each pencil formed of conics tangent at one point A, and interesting properties of these pencils are given. Finally, if both focal points describe curves, the congruence consists of a simple infinity of pencils as above, the point A describes a conic of the family, Γ, while the tangent of contact of the pencil envelops another conic Γ', bitangent to Γ at the two fixed points of the family if these are distinct and having contact of the third order with Γ if the fixed points coincide. This decomposition can be accomplished a second way, leading again to the same Γ and Γ'.

A. Schwartz (New York, N.Y.).

Lepskiĭ, M. M. A correspondence arising in the projection of curvilinear sets of points. Ukrain. Mat. Ž. **2** (1950), no. 3, 125–137. (Russian)

The concept of transformation of curvilinear sets of points by means of a curve of sliding ("Gleitkurve") was introduced into nomography by H. Schwerdt [Z. Angew. Math. Mech. **4** (1924), 314–323]. The present paper is a geometrical study of these transformations and of their products in the case that the curve of sliding reduces to a point.

D. J. Struik (Cambridge, Mass.).

Leps'kiĭ, M. M. On the convergence of a sequence of circuits. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 177–180. (Ukrainian. Russian summary)

Applications are made of the theory introduced in the paper reviewed above, among others to the generalization of Castillon's theorem by N. F. Četveruhin [Geometrical constructions and approximations, Učpedgiz, Moscow, 1935].

D. J. Struik (Cambridge, Mass.).

Leps'kiĭ, M. M. Convergence of circuits on boundaries of regions of direct and inverse monotonous convergence. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 106–109. (Ukrainian. Russian summary)

This paper, a further elaboration of the paper reviewed above, takes up certain questions of convergence, when the velocity is equal to one. This leads to questions of iteration, as discussed, e.g., by Isenkrahe [Das Verfahren der Functionswiederholung, Teubner, Leipzig, 1897] and certain cinematical problems introduced by M. A. Skuridin [Trudy Sem. Teorii Mašin i Mehanizmov **2** (1947), 55–100; MR **13**, 696].

D. J. Struik (Cambridge, Mass.).

★ **Pickert, Günter. Projektive Ebenen.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXX. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+343 pp. Geheftet DM 44.80; gebunden DM 48.60.

At the time of Hilbert's "Grundlagen der Geometrie" and Veblen and Young's "Projective geometry" [2 vols., Ginn, Boston, 1910, 1918] the existence of non-Desarguesian planes was recognized, but, after giving a few examples, the authors moved on to consideration of Desarguesian planes. A paper by Veblen and Wedderburn [Trans. Amer. Math. Soc. **8** (1907), 379–388] is perhaps the only exception to this general rule of ignoring non-Desarguesian planes. Interest in non-Desarguesian planes may be regarded as stemming from several sources. In 1928 a long series of papers on "Topologische Fragen der Differentialgeometrie" began, with Blaschke and Bol the principal authors. In a series of papers written in 1931–1935 Ruth Moufang founded the study of alternative planes. Baer made a study of collineations in 1942, and in 1943 the reviewer published a paper on "Projective planes" [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72]. More recently the interest in non-Desarguesian planes has been steadily increasing, there being at present groups actively engaged in these studies in Russia, Italy, Germany, and the United States.

The book under review is a very welcome synthesis of the body of research on projective planes, and the author is well qualified by his own researches to write such a book. The book is both complete and carefully organized.

The book begins with a discussion of incidence axioms and the theory of free extensions and free planes. The ternary operation is introduced for affine planes and their duals, and the properties of isotropy are given. Next comes a treatment of nets and the configurations of Reidemeister and Thomsen, together with their relationship to loops and what are called "double loops", related to 4-nets.

Chapters 3 and 4 are devoted to the Theorem of Desargues. The relationships to collineations are explored in detail, in particular to central collineations. In a

central collineation all the points of some line and all the lines through some point are fixed. The relationship to the reviewer's Theorem L and other special cases of the Theorem of Desargues is discussed in detail. The general theory of cross-ratios and the theorem of the complete quadrilateral is given.

The fifth chapter is devoted to the Theorem of Pappus. Hessenberg's result that the Theorem of Pappus implies the Theorem of Desargues is not given in its original synthetic form but in terms of arguments on perspectivities. Various conditions are shown to be equivalent to the Theorem of Pappus.

Chapters six and seven deal with alternative division rings and their equivalence to the validity of Theorem L. The Bruck-Kleinfeld theorem (proved independently by Skornyakov) that an alternative division ring is associative or a Cayley algebra is given. These chapters cover the material originally studied by Moufang.

Chapter eight deals with translation planes, these being affine planes with a transitive translation group. Such planes can be represented by a group, elements being points, and lines being given by appropriate subgroups and their cosets.

In Chapter nine, on ordered planes, various axioms of betweeness are considered together with their relationship to coordinates. The following chapter gives what little is known about topological projective planes. It is still not clear which axioms are most fruitful in topologizing planes.

Chapter eleven gives the Moufang theory of the relationship of Möbius nets to Desarguesian configurations of rank 8.

The final chapter is devoted to finite planes. It is pointed out that these are special cases of the symmetrical block designs. Essentially the Chowla-Ryser proof of existence conditions is given. It is not, however, observed that the Hasse-Minkowski results are needed for the converse. There is a brief discussion of difference sets and the cyclic projective and affine planes. The basic theorem on multipliers is given. The representation of finite planes by complete sets of orthogonal squares is given and a non-Desarguesian example of order 9. The proof by Pierce and the reviewer of the uniqueness of the plane of order 7 is given.

The only topic related to projective planes whose omission was noted by the reviewer was that of the Hjelmslev planes. Strictly these are not projective planes and the omission is surely justified. The reviewer would have preferred to see the names of the authors cited in the text along with the comparatively anonymous numbers referring to the bibliography, but this is purely a matter of taste. *Marshall Hall, Jr.* (Columbus, Ohio).

Ostrom, T. G. Ovals, dualities, and Desargues's theorem. Canad. J. Math. 7 (1955), 417–431.

An oval $\mathfrak{C}$ is a set of $n+1$ points no three of which are collinear, belonging to a graphic (or projective) plane π with $n+1$ points on each line. If π is Desarguesian and n is odd, then $\mathfrak{C}$ is a conic [cf. B. Segre, same J. 7 (1955), 414–416; MR 17, 72]; but there $\mathfrak{C}$ is studied without assuming π to be a Desarguesian plane.

A line of π, considered with respect to $\mathfrak{C}$, can only be of one of three types, i.e. a secant, a tangent (or absolute), or an exterior line, according as it contains 2, 1, or 0 points of $\mathfrak{C}$. Likewise, if n is odd, a point of π can only be of one of three types, i.e. an exterior, an absolute, or an interior point, according as it lies on 2, 1, or 0 tangents of $\mathfrak{C}$. The oval $\mathfrak{C}$ thus defines a sort of quasi-polarity, a priori restricted to exterior points and secants, and to absolute points and lines, related in an obvious way.

The present paper studies $\mathfrak{C}$, first in the hypothesis that there is a non-identical collineation σ of π transforming $\mathfrak{C}$ in itself and such that, excepting the identity, no power of σ leaves fixed any points which are not left fixed by σ itself. Among other results, it is shown that C determines a polarity if, for every non-absolute point U of π, there is a collineation taking $\mathfrak{C}$ into itself which leaves U and every point on some line u fixed (u depending on U). Then certain other respects are pointed out in which an oval resembles a conic in a Desarguesian plane, especially in its relations to a properly defined type of harmonic sets and in the case where the oval consists of the absolute points of a polarity. Hence ovals are studied in cyclic and other transitive planes, finding that Desargues's configuration comes up again and again for suitably restricted pairs of triangles. Finally there is shown the existence of Fano's configuration (i.e. of complete quadrangles with collinear diagonal points) for certain non-Desarguesian planes having n even. *B. Segre* (Rome).

Bruck, R. H. Recent advances in the foundations of euclidean plane geometry. Amer. Math. Monthly 62 (1955), no. 7, part II, 2–17.

Verf. skizziert einen Ausschnitt aus der Weiterentwicklung der Hilbert'schen Grundlagenforschung, die den Zusammenhang zwischen geometrischen Systemen und algebraischen Strukturen betrifft. Er berücksichtigt dabei insbesondere die Algebraisierung der Eulidischen Ebene mit Hilfe eines ternären Ringes nach M. Hall [Trans. Amer. Math. Soc. 54 (1943), 229–227; MR 5, 72] und den stufenweisen Ausbau des ternären Ringes zu einem alternativen Divisionsring aufgrund der zusätzlichen Forderung an die Euklidische Ebene, dass in ihr spezielle Desargues'sche Sätze gelten. Abschliessend berichtet Verf. über die wichtigen Resultate in der Theorie der alternativen Ringe, die in den letzen 5 Jahren gewonnen wurden. *R. Moufang* (Frankfurt a.M.).

Hall, Marshall, Jr. Finite projective planes. Amer. Math. Monthly 62 (1955), no. 7, part II, 18–24.

Aus der Entwicklung der letzten 55 Jahre hebt Verf. 27 Arbeiten heraus, die sich um zwei Methoden zur systematischen Konstruktion endlicher projektiver Ebenen gruppieren, wobei insbesondere die nicht-Desargues'schen Typen interessieren: 1) auf algebraischem Wege durch generelle Einführung von Koordinaten mit Hilfe eines ternären Ringes; 2) auf kombinatorischem Wege durch Aufsuchung von paarweise orthogonalen lateinischen Quadraten aus n Buchstaben. Ferner geht Verf. ein auf die Konstruktion von endlichen projektiven Ebenen aus einer Kollineationsgruppe, was auf zahlentheoretische Probleme führt. *R. Moufang* (Frankfurt a.M.).

Wesson, J. R. Finite plane projective geometries. Amer. Math. Monthly 62 (1955), no. 7, part II, 32–40.

Ziel des Verf. ist, die kombinatorische Konstruktion endlicher projektiver Ebenen durch Zurückführung auf eine algebraische Struktur durchsichtiger zu machen. Im Anschluss an seine Thesis [Vanderbilt Univ., Nashville, Tenn., 1953] beschreibt Verf. zunächst die sämtlichen Inzidenten in einer endlichen projektiven Ebene dadurch, dass er die Punkte von 1 bis n^2-n+1 durchnumeriert und sie zu je n auf die n^2-n+1 Geraden verteilt; hierbei werden zuerst alle Punkte auf den n Geraden durch den

Punkt 1 ausgezählt, dann die Punkte auf den Geraden durch den Punkt 2 u.s.w. Dies gibt Anlass zur Betrachtung von bestimmten Permutationen der Elemente 0, 1, 2, $\cdots$, $n-2$, die sich als eine ternäre Operation, angewandt auf diese Elemente, beschreiben lässt und auf den Begriff eines sogenannten H-Systems führt: H-System ist eine endliche Menge von Elementen 0, a, b, $\cdots$, zwischen denen eine ternäre Operation definiert ist mit den Eigenschaften: 1) $a0b=0ab=b$, $1a0=a10=a$; 2) $abx=c$ hat genau eine Lösung x; 3) für $a\neq c$ haben $axy=b$, $cxy=d$ genau eine Lösung x, y. Die Zuordnung zwischen einer endlichen projektiven Ebene und einem H-System ist ein-eindeutig. Die Bedingung dafür, dass eine endliche Ebene existiert mit r Punkten auf einer Geraden, wenn r keine Primzahl ist, lässt sich so formulieren, dass ein H-System existiert, das entweder „nicht-associativ" oder „nicht-kommutativ" ist. Diejenigen H-Systeme, in denen die durch $a+b=1ab$, $ab=ab0$ erklärten binären Verknüpfungen einen Körper bilden, lassen sich charakterisieren. Für den Fall, dass die Anzahl der Elemente des Systems eine Primzahl ist, ist die Gültigkeit des „associativen" und „kommutativen" Gesetzes hinreichend und notwendig. Man vergleiche hierzu die von M. Hall [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72] angegebenen Postulate eines ternären Systems zur Charakterisierung projektiver Ebenen und die oben ref. Arbeit desselben Verf. *R. Moufang* (Frankfurt a.M.).

Ryser, H. J. Geometries and incidence matrices. Amer. Math. Monthly **62** (1955), no. 7, part II, 25–31.

An Hand eines Literaturverzeichnisses von 22 Arbeiten aus den Jahren 1900–1954 gibt Verf. eine gedrängte, reichhaltige Übersicht über gelöste und ungelöste Probleme, die in dem Studium der endlichen projektiven Ebenen ihren Ausgangspunkt und ihr Ziel haben, aber in ihrer allgemeinen kombinatorischen Formulierung auch andere mathematische Disciplinen berühren. Das zugrunde liegende kombinatorische Problem ist die Auffindung von v, k, λ Konfigurationen: v Elemente x_i sind so auf v Klassen T_j zu verteilen, dass jede Klasse genau k verschiedene Elemente enthält und je zwei Klassen genau λ Elemente gemein haben ($0<\lambda<k<v$). Zu jeder solchen Konfiguration gehört eine Inzidenzmatrix aus v Zeilen und v Spalten, deren Elemente a_{ij} den Wert 1 oder 0 haben, je nachdem x_j zu T_i gehört oder nicht. Speziell für $\lambda=1$, $k=n+1$, $n\geqq 2$ beschreibt die Konfiguration eine endliche projektive Ebene mit $n+1$ Punkten auf jeder Geraden. Es ist stets $AA^{\mathrm{T}}=B$, wobei die Elemente der Hauptdiagonale von B gleich k, alle übrigen Elemente von B gleich λ sind. Weitere Untersuchungen knüpfen an diese Matrixgleichung an, wenn B in der genannten Form gegeben ist und die Matrix A mit rationalen bezw. ganzen Elementen gesucht ist. *R. Moufang.*

Gans, David. An introduction to elliptic geometry. Amer. Math. Monthly **62** (1955), no. 7, part II, 66–75.

van der Vaart, H. R. The content of some classes of non-Euclidean polyhedra for any number of dimensions, with several applications. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 199–209, 210–221; errata 564.

This is a full account of the work briefly announced in Experientia **9** (1953), 88–90 [MR **14**, 1007] on the content of a polyhedral cone or spherical polytope. Like Schläfli [Ges. Math. Abh., Bd. 1, Birkhäuser, Basel, 1950, pp. 234–235; MR **11**, 611], the author reduces the problem to the case of an orthoscheme: the analogue of a right-angled triangle. Like Schläfli again, he expresses the result both as a multiple integral and by a differential equation. He saves some space by using symbols for matrices. *H. S. M. Coxeter* (Toronto, Ont.).

Mihăileanu, N. Nichteuklidische Geometrie. Editura Academiei R. P. Romîne, Bucarest, 1954. 143 pp. (Romanian)

Verf. hält es für eine erste Einführung in die nichteuklidische Geometrie für zweckmäßig, die grundlegenden Ergebnisse auf verschiedenen Wegen zu erarbeiten. Zum Eingang scheint ihm die Betrachtung der Geometrie auf der Kugel wegen der vielen Analogien zur hyperbolischen Geometrie geeignet. Somit behandelt er im ersten der fünf Teile die sphärische Geometrie bis zu den sphärischen Kegelschnitten. Der zweite Teil behandelt vom projektiven Standpunkt das Absolute, die hyperbolische Geometrie, elliptische und hyperbolische Trigonometrie, nichteuklidische Bewegungen. Der dritte Teil bringt die analytische Geometrie der hyperbolischen Ebene und des Raumes. Im vierten Teil wird die Differentialgeometrie (Linien- und Flächenelement, Krümmung, Geodätische, Frenetsche Formeln, Pseudosphäre, Riemannsche Räume) behandelt. Der letzte Teil bringt die Modelle von Poincaré und Klein und die Hilbertsche Axiomatik der nichteuklidischen Geometrie.

M. Zacharias (Zbl **56**, 138).

Convex Domains, Extremal Problems, Distance Geometry

★ **Hadwiger, H. Altes und Neues über konvexe Körper.** Birkhäuser Verlag, Basel und Stuttgart, 1955. 116 pp. 13.50 Swiss francs.

A convex body A, B, $\cdots$ in the sense of this book is a bounded closed convex subset of euclidean 3-space. Starting with few prerequisites the present book concentrates on some developments of the theory of the volume V, the surface F and the mean curvature integral M of A. Apart from the last chapter, the methods are mostly direct.

The first chapter introduces on 16 pp. the basic concepts such as the Minkowski addition $A\times B$, inner and outer parallel bodies A_ϱ [$\varrho\geqq -r$; r=radius of the insphere], real-valued functionals $X(A)$. The space $\mathfrak{K}$ of the A's is metrized and its local compactness is proved [Blaschke's "Auswahlsatz"].

Each A can be approximated by convex polyhedrons P. Each Steiner symmetrization S is a continuous mapping of $\mathfrak{K}$ into itself. Similarly, $A\times B$ depends continuously on A and B. Let A be fixed. Then the closures in $\mathfrak{K}$ both of the set $\mathfrak{S}$ of all B's obtained from A by a finite number of S's and of the set of all finite Minkowski sums $k^{-1}A^{\delta_1}\times k^{-1}A^{\delta_2}\times\cdots\times k^{-1}A^{\delta_k}$ contain spheres [The δ_i indicate rotations about the origin [Chapter II, 7 pp.].

The functionals $V(A)$, $F(A)$, $M(A)$ are definite [$V(A)\geqq 0$, $\cdots$], monotonic [$B\subset A$ implies $V(B)\leqq V(A)$, $\cdots$] and homogeneous [$V(\lambda A)/V(A)$ etc. are powers of $\lambda>0$]. Steiner's formula and these properties are first verified for P's then extended to A's. The set $\mathfrak{M}$ [$\mathfrak{R}$] of all linear combinations (1) $X(A)=\alpha_0+\alpha_1 M(A)+\alpha_2 F(A)+\alpha_3 V(A)$ with real [non-negative] coefficients consists of all

continuous [of all monotonic] $X(A)$ which satisfy (i) $X(A)=X(B)$ if A and B are congruent, (ii) $X(A\cup B)+X(A\cap B)=X(A)+X(B)$ if the convex body $A\cup B$ is divided by a plane into A and B. $V(A)$ is described in a similar way. As an application, Cauchy's integral formulas are proved [Chapter III, 20 pp.].

The inequalities for V, F, M are discussed in Chapter IV (33 pp.). The quoted theorem on the set $\mathfrak{S}$ implies $V(A\times B)^{\frac{1}{3}}\geqq V(A)^{\frac{1}{3}}+V(B)^{\frac{1}{3}}$. This implies the Brunn-Minkowski Theorem which in turn yields Minkowski's inequalities (2) $F^3\geqq 3MV$ and (3) $M^3\geqq 4\pi F$. They imply the isoperimetric inequality (4) $F^3\geqq 36\pi V^2$. Except for (2) where the reader is referred to Bol's original paper, the conditions for equality to hold are discussed. (3) is also proved by integral geometry. The Brunn-Minkowski Theorem is extended to concave families of convex bodies and the family of all A_ϱ's $[-r\leqq\varrho<\infty]$ is shown to be concave. For every $\varrho\geqq -r$, $dV(A_\varrho)/d\varrho=F(A_\varrho)$. A discussion of the inner parallel bodies leads to a simple improvement of (4). Bol and Knothe's refinements of (3) and (4) by means of certain norm bodies are obtained. Following M. Goldberg, (4) is improved for P's. This chapter closes with a detailed discussion of Blaschke's Diagram both for $\mathfrak{R}$ and for some subsets of $\mathfrak{R}$.

The last chapter is an introduction to the integral geometry of convex bodies (20 pp.). Various results of the following type are proved: Let A [A_0] be variable [fixed]. If Φ belongs to a suitable class of functionals, then $\int\Phi(AA_0)d*$ or $\int\Phi(A)d*$ extended over suitable groups of motions is a functional $X(A)\subset\mathfrak{R}$. Hence (1) applies where the α_i's depend only on Φ. Numerous specializations include Crofton's and Cauchy's formulas and the "Kinematische Hauptformeln" of Blaschke and Santaló.

The selected subject matter reflects the author's interests and achievements. Once it is accepted, the reader can but admire its skillful treatment. The reviewer found the book readable and stimulating. Each chapter closes with bibliographical notes. A 13 pp. bibliography is at the end. *P. Scherk* (Saskatoon, Sask.).

Hadwiger, H. Eulers Charakteristik und kombinatorische Geometrie. J. Reine Angew. Math. **194** (1955), 101–110.

Let K be the ring of all convex bodies in either a Euclidean or spherical finite-dimensional space. It is shown there exists an additive integer-valued functional f which applied to a convex polyhedron P yields the familiar alternating sum of the Euler-Poincaré characteristic (whose value here is of course 1). Each element k of K is a finite union of convex bodies whose intersections determine a nerve of k. Elements are equivalent if their nerves are isomorphic. f is invariant under equivalence and is a tool for investigating the intersection relations of families of convex sets. The paper exploits this tool and contains some old and some new results more or less on the pattern of Helly's theorem. *D. G. Bourgin.*

Ghika, Al. Ensembles extrémaux et hyperplans d'appui maximaux. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 59–64. (Romanian. Russian and French summaries)

Some elementary properties of convex polyhedra and of extremal subsets of convex sets are established. A supporting hyperplane H of a (closed) convex body A is called "maximal" provided no other supporting hyperplane contains $H\cap A$, and it is claimed erroneously that each boundary point x of a convex body must be contained in a maximal supporting hyperplane. (The assertion fails if A is strictly convex and is supported at x by more than one hyperplane; in particular, in the case of a biconvex lens.) [In a separable Banach space, each convex body must admit a maximal supporting hyperplane. However, there exist separable noncomplete normed linear spaces and nonseparable Banach spaces whose unit cells admit no maximal supporting hyperplane.] *V. L. Klee, Jr.* (Los Angeles, Calif.).

Mahler, Kurt. The p-th compound of a sphere. Proc. London Math. Soc. (3) **5** (1955), 385–391.

Let Y be a vector in R_N, $N=\binom{n}{p}$, the components of which are the determinants of the $p\times p$ matrices in the matrix $[X_1,\cdots,X_p]$, where $X_1,\cdots,X_p$ are vectors from R_n. Γ_n^p denotes the pth compound of the unit n-sphere G_n, i.e., the convex hull of all points Y for which $X_1,\cdots,X_p\in G_n$. The author shows $\Gamma_n^p=G_N\cap\Omega(n,p)$, where G_N is the unit sphere in R_N and $\Omega(n,p)$ is the Grassmann manifold in R_N. It follows from this that Γ_n^p is the convex hull of the intersection of the surface of G_N with $\Omega(n,p)$. The second of these results is used to construct the support and distance functions of Γ_4^2. From the latter it follows that Γ_4^2 consists of all points $Y=(\xi_1,\cdots,\xi_6)$ for which $\sum_{H=1}^{3}(\xi_H+\xi_{H+3})^2\leqq 1$, $\sum_{H=1}^{3}(\xi_H-\xi_{H+3})^2\leqq 1$. *D. Derry.*

Seidel, J. Angles and distances in n-dimensional euclidean and noneuclidean geometry. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 329–335, 336–340, 535–541.

These three notes give a unified treatment of certain metric aspects of n-dimensional euclidean, hyperbolic, elliptic, and spherical geometry. This is done by considering a basic real linear space, for each two points x, y of which an inner product (x, y) is defined. The condition $(x, x)>0$, $x\neq 0$, leads to elliptic and spherical spaces, while a weaker condition defines two types of spaces that give rise to the four geometries. The notion of inner product is basic. [Reviewer's note. It would seem that perhaps a more elegant treatment might be obtained by assuming as underlying space a quasi-inner-product space [see, e.g., Blumenthal, Amer. J. Math. **72** (1950), 686–698; MR **12**, 436.] *L. M. Blumenthal.*

See also: Ehrhart, p. 350; Sawyer, p. 351; Ghika, p. 386; Fan and Glicksberg; p. 386.

Differential Geometry

Amsler, Marc-Henri. Des surfaces à courbure négative constante dans l'espace à trois dimensions et de leurs singularités. Math. Ann. **130** (1955), 234–256.

Although it is possible to locally imbed in three-dimensional euclidean space E^3 a surface of constant Gaussian curvature equal to -1, Hilbert [Grundlagen der Geometrie, 5th ed., Teubner, Leipzig, 1922, Appendix] proved that it is impossible to imbed the hyperbolic plane in E^3. The author studies the nature of the singularities which cause this failure.

A surface F of curvature -1 is said to have no prolongation if it cannot be considered as a subset of a surface F' also of curvature -1. A point S, which may or may not be on F, is called a singularity of F if there exists a sequence of points $P_0, P_1, \cdots$ of F which converge

to S in E^3 with the property that the distance between P_0 and P_i as i goes to infinity is bounded and there exists no prolongation of F which contains S in its interior. In addition the author introduces the concept of a ramification point which is too complicated to define here. The main result is the following: Every non-prolongable surface of class at least C^3 with curvature -1 and without ramification points has on its boundary at least one curve formed exclusively of singular points. The author further shows that if the surface is analytic it can have no ramification points. In addition, in the analytic case we find a very detailed study of the singular curves. The paper is also well-supplied with many examples.

L. Auslander (Princeton, N.J.).

Ara, Rahmat, and Pinl, M. The ideal straight lines on Scherk's minimal surface. Pakistan J. Sci. Res. **5** (1953), 145–149.

By using complex, projective coordinates $x=x_1/x_0$, $y=x_2/x_0$, $z=x_3/x_0$ and by discussing the limit behavior for $x_0\to 0$, the authors determine the intersection of Scherk's minimal surface $e^z=\cos y/\cos x$ with the ideal plane $x_0=0$. The intersection consists of nine straight lines: $x_3=0$, $x_2=\pm ix_1$, $x_3=\pm ix_2$, $x_3=\pm i(x_1-x_2)$ and $x_3=\pm i(x_1+x_2)$, and they have all together 20 points of intersections. It is then proved that each of the three pairs of lines ($x_1=0$, $x_2=0$), ($x_3=\pm i(x_1-x_2)$) and ($x_3=\pm i(x_1+x_2)$) on the plane $x_0=0$, gives two families of parallel planes which cut the minimal surface in a conjugate net, consisting of translation curves.

Y. W. Chen (Zbl **52**, 384).

Arghiriade, Em. Sur les quadriques osculatrices d'une surface. Com. Acad. R. P. Române **3** (1953), 19–23. (Romanian. Russian and French summaries)

From the quadrics which have contact of order 2 with a surface S at a point x the author selects certain pencils whose members cut S along curves which have triple points at the origin, the tangents at this point determining a triad. If we restrict the choice of the pencils further by requiring that the triad be a perfect cube, we get three "pencils of coincidence". A line d passing through x will have a polar with respect to each pencil of coincidence. It is proved that these polars are the inflectional tangents of the antihessian of the Segre cubic of d.

Using the concept of differential conjugate, the author explains the O. Mayer relationship for two pencils of osculating quadrics. If two pencils have this relationship, the polars of a line d through x with respect to these pencils will correspond in the homology which transforms the Segre cubic of d into its hessian. *A. Schwartz.*

Arghiriade, Em. Sur la polarité par rapport aux quadriques osculatrices d'une surface. Com. Acad. R. P. Române **3** (1953), 179–185. (Romanian. Russian and French summaries)

Let C_3 be the cubic of Segre corresponding to a line d. Let H_1 be the antihessian of C_3 and let Γ_1 be the cayleyian of H_1. Then Γ_1 is the envelope of the polars of d with respect to the quadrics of B. Gambier. [J Math. Pures Appl. (9) **15** (1936), 151–162]. The quadrics are redefined here as osculating quadrics whose tangential triads admit hessians formed with two conjugate lines.

Let H_2 be the antihessian of H_1 and Γ_2 its cayleyian. Then Γ_2 is the envelope of the polars of d with respect to the osculating quadrics which cut the surface along a tangential triad containing two conjugate lines. The inflectional tangents of H_2 are the polars of d with respect to certain osculating quadrics whose tangential triads are formed from the Segre direction counted twice and the Darboux directions. *A. Schwartz* (New York, N.Y.).

Arghiriade, E. Sur les correspondances de Cech. Com. Acad. R. P. Române **4** (1954), 5–14. (Romanian. Russian and French summaries)

Among the theorems demonstrated are the following: Let d be a line passing through point x of a surface S and d' its transform in the fundamental polarity. Let C_0 and Γ_0 be the cubic curve and the cone which correspond to d, d' in the general Čech correspondence, and let C_i and Γ_i be their hessians of order i (antihessians if i is negative). Then the osculating conics at x to the two branches of the cubic C_i correspond by polar reciprocity with respect to the Lie quadric to the asymptotic cones of Γ_{i-1} and the asymptotic conics at x of the cubic C_i correspond to the osculating quadric cones of Γ_{i+1}.

If C_0 is the cubic which corresponds to d in the Segre correspondence, the osculating conics at x to the two branches of c_0 intersect at x and at three other points. The lines joining these other points are the polars of d with respect to the quadrics of Gambier which have a Segre direction as generating line.

The osculating planes of the Darboux lines form a trihedral whose edges are the polars of the second axis of Čech with respect to the same Gambier quadrics.

A. Schwartz (New York, N.Y.).

Arghiriade, E. Sur la correspondance de Segre. Com. Acad. R. P. Române **4** (1954), 97–100. (Romanian. Russian and French summaries)

Let C be the Segre cubic corresponding to a line d passing through a point x of a surface S. Let δ be the generating line of a Gambier pencil of quadrics. Let δ meet C at a point N. Draw the tangents to C which pass through N. The line joining the points of contact of these tangents is the polar of d with respect to the pencil of Gambier quadrics. At each point of inflection of C one can draw one tangent to C different from the tangent of inflection. These tangents are the transforms of coincidence of d. (See the paper reviewed third above.)

A. Schwartz (New York, N.Y.).

Backes, F. Sur une extension de la notion d'asymptotique d'une surface. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 723–733.

Dans un article antérieur [même Bull. (5) **40** (1954), 118–124; MR **15**, 825], l'auteur a étendu la notion de réseau conjugué en substituant au plan tangent en M à une surface S une sphère de rayon $\varrho(u, v)$ tangente en M à S, u et v étant les paramètres curvilignes sur S. Il a désigné par A_ϱ le réseau associé à un choix déterminé du rayon $\varrho(u, v)$. L'article actuel traite d'une extension analogue concernant les asymptotiques de S. L'auteur est ainsi conduit à de nouveaux réseaux dont les tangentes sont celles des cercles cararactéristiques des sphères de courbure normale. Entre autres propriétés, il rattache à ces nouveaux réseaux l'étude d'une correspondance entre deux surfaces dans laquelle les lignes de courbure se correspondent. Et la considération des cercles possédant des sphères focales et normaux à deux surfaces (au moins) le conduit, pour ces cercles, à la propriété caractéristique suivante: A la sphère touchant les surfaces aux points où elles coupent le cercle, correspondent des réseaux A_ϱ homologues sur les surfaces envisagées. *P Vincensini.*

Marcus, F. Sur une congruence W appartenant à un complexe linéaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 837–838.

Dans une note récente [même Bull. (5) **40** (1954), 983–989; MR **16**, 513], L. Godeaux, a montré que la congruence W envisagée par lui appartient à un complexe linéaire. L'auteur de la note actuelle retrouve ce résultat, en observant que les deux nappes focales de la congruence sont transformées l'une de l'autre dans une corrélation, et en invoquant un théorème de G. Fubini. *P. Vincensini*

Teleman, C. Les transformations qui laissent invariant le groupe orthogonal ou le groupe symplectique. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 355–363. (Romanian. Russian and French summaries)

The most general transformations $y^i=\varphi^i(x^1, \cdots, x^n)$, $i, j=1, 2, \cdots, n$, which leave the complete group of rotations in n variables $G_{n(n-1)/2}$ invariant are given by $y^i=u\alpha_j{}^i x^j$, where $\alpha_j{}^i$ is an orthogonal matrix with constant terms and u an arbitrary function of $(x^1)^2+(x^2)^2+\cdots+(x^n)^2$. These transformations and

$$x^{2h+1}=\bar{u}(y)[y^{2h-1}\cos\theta(y)-y^{2h}\sin\theta(y)],$$

$$x^{2h}=\bar{u}(y)[y^{2h-1}\sin\theta(y)+y^{2h}\cos\theta(y)],$$

$$y=[(y^1)^2+\cdots+(y^{2p})^2]^{\frac{1}{2}},\ h=1, \cdots, p,\ n=2p,$$

are the only ones which leave invariant the unitary group

$$X_{2h-1};\ X_{2h-1,2k-1}+X_{2h,2k};\ X_{2h-1,2k}-X_{2h,2k-1},$$
$$X_{i,j}=x^i\partial_j f-x^j\partial_i f,\ h, k=1, \cdots, p.$$

D. J. Struik (Cambridge, Mass.).

Libermann, Paulette. Sur la semi-involution. C. R. Acad. Sci. Paris **241** (1955), 1444–1447.

Let G be a p-dimensional subgroup of the real general linear group L_n. Then the author shows that either G can be prolonged to give an involutive group or after a finite number of prolongations the prolonged groups will be zero. The author further obtains some numerical relations between the dimensions of the prolonged groups and the characters of G. *L. Auslander* (Princeton, N.J.).

Eells, James, Jr. Geometric aspects of currents and distributions. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 493–496.

The author sketches a direct geometric procedure on which a theory of currents on a C^∞ manifold can be based. From $C_r(M)$, the linear space of C^∞ singular r-chains on M, one passes to a space $\mathfrak{C}_r(M)$ by forming the direct limit of a system of spaces formed by completing $C_r(M)$ with respect to a certain sequence of norms. Let $\mathfrak{C}^r(M)$ be the inverse limit of the corresponding conjugate spaces. The first theorem asserts that for $X \in \mathfrak{C}^r(M)$, there is a unique C^∞ r-form ξ whose derivatives satisfy a certain boundedness condition and is such that for any r-simplex on M, $X\cdot\sigma=\int_\sigma \xi$. A second theorem gives for fixed $A \in \mathfrak{C}_r(M)$ and arbitrary $X \in \mathfrak{C}^r(M)$, a representation of $X\cdot A$ as an integral over M of an expression which involves a finite set of summable vector fields and r-vector fields depending only on A. Cup and cap products can be defined in $\mathfrak{C}^r$, $\mathfrak{C}^r$; they correspond thru the theorems stated above, to exterior and interior products of r-forms and s-vector fields on M. The methods used lean heavily on methods used by H. Whitney in a book under preparation.

P. A. Smith (New York, N.Y.).

See also: Chevalley, p. 342.

Riemannian Geometry, Connections

★ **Yano, Kentaro. Gruppi di trasformazioni in spazi geometrici differenziali.** Istituto Matematico, Roma, 1953–54. 281 pp. (mimeographed).

This is an expository account of groups of transformations operating on various spaces studied in differential geometry. The first four chapters give a concise account of Riemannian geometry, spaces with an affine connexion, projective geometry of paths and conformal Riemannian geometry. A brief account of groups of transformations is contained in chapter V where the treatment is similar to that in Eisenhart's standard book. The next two chapters deal with motions in a Riemannian space, affine motions, affine collineations and with Lie derivation. Chapter VIII considers properties of Riemannian spaces which admit groups of motions with a certain number of parameters – for example, it is proved that if an n-dimensional Riemannian space, with $n>2$, $n\neq 4$, admits a group of motions with a number of parameters exceeding $\frac{1}{2}n(n-1)+1$, then the space has constant curvature. The following three chapters contain similar results for the group of affine transformations, projective collineations and conformal transformations.

Each of the eleven chapters in this book is restricted to local properties, and reference to global properties is confined to an appendix of 14 pages which gives a brief account of complex analytic manifolds, Hermitian metrics, Kähler metrics, almost complex manifolds and pseudo-Kählerian manifolds. Conditions are given for the non-existence of pseudo-analytic covariant and contravariant vectors on compact pseudo-Kählerian manifolds.

T. J. Willmore (Liverpool).

Ispas, C. I. Au sujet des dérivées de Lie et de la déformation des vecteurs contrevariants des espaces à connexion. Com. Acad. R. P. Romîne **5** (1955), 479–488. (Romanian. Russian and French summaries)

In a study of K. Yano, Groups of transformations in generalized spaces [Akad. Press, Tokyo, 1949; MR **10**, 481] the following operators are introduced

$$\sum_\alpha \begin{vmatrix} \frac{\partial}{\partial x^\alpha} & 0 \\ 0 & \xi^\alpha \end{vmatrix},\ \sum_\alpha \begin{vmatrix} \frac{\partial}{\partial x^\alpha} & 0 \\ 0 & \xi^\beta \end{vmatrix},\ \sum_\alpha \begin{vmatrix} \nabla_\alpha & 0 \\ 0 & \xi^\alpha \end{vmatrix},\ \sum_\alpha \begin{vmatrix} \nabla_\alpha & 0 \\ 0 & \xi^\beta \end{vmatrix}.$$

The Yano deformation is decomposed in the form

$$Dv^\lambda=(D_1-D_2)v^\lambda,\ D_1v^\lambda=\frac{\partial v^\lambda}{\partial x^\alpha}\xi^\alpha\cdot\delta t,\ D_2v^\lambda=\frac{\partial \xi^\lambda}{\partial x^\alpha}v^\alpha\cdot\delta t,$$

which decomposition is illustrated by a diagram.

D. J. Struik (Cambridge, Mass.).

Malgrange, Bernard. Formes harmoniques sur des espaces de Riemann à ds^2 analytique. C. R. Acad. Sci. Paris **240** (1955), 1958–1960.

The author announces two theorems valid for a class of differential operators on real analytic manifolds. He illustrates his methods by proving the theorems for the Laplace operator. Theorem 1. Let Ω be a real analytic Riemannian manifold which is orientable, connected, and open. Then Δ maps the class of C^∞ functions (respectively distributions) onto itself. Theorem 2. Let O be an open subset of Ω. A necessary and sufficient condition that every function which is harmonic in O should be, on every compact subset of O, the uniform limit of functions which are harmonic in Ω is that the complement

of O have no connected compact components. He states that corresponding theorems are valid for the operator $d_{\bar{z}}$ on a Riemann surface, Δ for differential forms and currents, and certain elliptic systems considered by Petrowsky and John. A corollary of the second case is that in every class of d-homologous forms there exists a harmonic form. Finally, he states that his results lead to the existence of elementary solutions which are globally defined. *M. P. Gaffney* (Evanston, Ill.).

Blum, Richard. Subspaces of Riemannian spaces. Canad. J. Math. **7** (1955), 445–452.

This paper is concerned with some aspects of the local imbedding problem for a Riemann space V_n in a Riemann space V_N. As is well known, a set of necessary conditions for this imbedding is given by the equations of Gauss, Codazzi and Kuehne. Beginning with equations derived from the above and analogous to those first used by Allendoerfer [Amer. J. Math. **61** (1939), 633–644; MR **1**, 28] for subspaces of a Euclidean space, the author shows how it is possible to extend some of his earlier results of spaces V_n imbedded in a Euclidean space E_N [Bull. Math. Soc. Roumaine Sci. **47** (1946), 144–201; **48** (1947), 88–101; C. R. Acad. Sci. Paris **224** (1947), 708–710; MR **9**, 618; **11**, 399; **9**, 66] to the present case. In this way, according to the rank of certain matrices, an exact count is given of the number of Codazzi-Kuehne equations which are a consequence of the remaining ones and the Gauss equations. The special case where the surrounding space V_N is a space of constant curvature with dimension number $N=n+1$ is considered in greater detail. In this case, the Gauss-Codazzi equations are both necessary and sufficient for the local imbedding of V_n. Generalizing the type number τ first defined by T. Y. Thomas [Acta Math. **67** (1936), 169–211] for hypersurfaces of a Euclidean space, the author proves that according as $\tau=2$, 3, at least 4, the number of independent Codazzi equations is $3n-4$, 5, 0 respectively while the remaining Codazzi equations are a consequence of these independent ones and the Gauss equations. The case $\tau=0$ is trivial and $\tau=1$ is impossible. *A. Fialkow* (Brooklyn, N.Y.).

Suguri, Tsuneo, and Nakayama, Shigeru. Note on Riemannian spaces and contact transformations. Tensor (N.S.) **5** (1955), 1–16.

This note is concerned with the problem of obtaining a Finsler space from a Riemannian space by applying a contact transformation. The authors point out that whereas Eisenhart [Ann. of Math. (2) **49** (1948), 227–254; MR **9**, 380] was concerned only with sufficient conditions in order that such a contact transformation can exist, they study both the necessary and sufficient condition. Much of the discussion is concerned with the contact frames, which are slightly more general than those used by Yano and Davies [Ann. Mat. Pura Appl. (4) **37** (1954), 1–36; MR **16**, 626]. *E. T. Davies* (Southampton).

Egorov, I. P. Maximally mobile Riemannian spaces V_4 of nonconstant curvature. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 9–12. (Russian)

Dans ce mémoire on détermine tous les espaces V_4 de Riemann à quatre dimensions, qui sont les plus mobiles (maximalement mobile) et possèdent les courbures nonconstantes. Ces espaces sont nécessairement les espaces d'Einstein avec le groupe $\mathfrak{G}_\infty$ des mouvements. L'auteur démontre les théorèmes suivants. Théorème 1: Il existe deux et seulement deux espaces différents à quatre dimensions qui sont les plus mobiles et possèdent les courbures nonconstantes et pour lesquels l'élément linéaire est définit par la formule

$$(1)\quad ds^2=\frac{(\varepsilon+\sum_1^4 y^{i^2})\sum_1^4 dy^{i^2}-(\sum_1^4 y^i dy^i)^2-(y^1dy^2-y^2dy^1+y^3dy^4-y^4dy^3)^2}{(y^{1^2}+y^{2^2}+y^{3^2}+y^{4^2}+\varepsilon)^2}$$

$$(\varepsilon=\pm 1).$$

Le groupe $\mathfrak{G}_\infty$ des mouvements est compact pour $\varepsilon=+1$ et noncompact pour $\varepsilon=-1$ (y^1, y^2, y^3, y^4 sont les coordonnées normales dans un environ du point $M\in V_4$). Le second théorème donne la signification géométrique des espaces mentionnés. Si l'on pose dans la formule (1) $x^1=y^1+iy^2$, $x^2=y^3+iy^4$, $x^3=\overline{x^1}$, $x^4=\overline{x^2}$ on obtient

$$(2)\quad ds^2=e^{-2v}[\varepsilon(dx^1dx^3+dx^2dx^4)+(x^3dx^1-x^1dx^2)\,(x^4dx^3-x^3dx^4)],$$

où $e^v=x^1x^3+x^2x^4+\varepsilon$. Alors on a le Théorème 2: Les plus mobiles espaces V_4 de Riemann avec les courbures nonconstantes sont les réprésentations réelles de l'espace des couples point-droite du plan projectif complexe, dont les points et les droites sont harmoniques par rapport à la quadrique de Hermite

$$\varepsilon+x^1\overline{x^1}+x^2\overline{x^2}=0.$$

Le travail accompagne les résultats de G. Vranceanu [Publ. Math. Debrecen **3** (1954), 24–32; MR **15**, 897] et aussi le théorème de Fubini [Ann. Mat. Pura Appl. (3) **9** (1903), 33–90]. *F. Vyčichlo* (Prague).

Tsien, T. C. Über die Formeln von Dini und Levi-Cività der geodätischen Abbildungen. J. Chinese Math. Soc. **2** (1953), 144–156. (Chinese. German summary)

Beweis der folgenden Sätze: 1. Wenn zwischen zwei Flächen eine Abbildung besteht, welche die beiden Scharen isotroper Linien der ersten Fläche auf zwei Scharen komplexer geodätischer Linien der zweiten Fläche und zugleich die beiden Scharen isotroper Linien der zweiten Fläche auf zwei Scharen komplexer geodätischer Linien der ersten Fläche abbildet, so ist die Abbildung geodätisch. Die Fundamentalformen der beiden Flächen lassen sich auf die klassische Dinische Form bringen. 2. Angenommen zwei Riemannsche Räume haben ein gemeinsames orthogonales Bezugssytem und es besteht eine Abbildung der beiden Räume aufeinander, welche ein gewisses System von 2^{n-1} isotropen Kurvenkongruenzen des ersten Raumes auf 2^{n-1} komplexe geodätische Normalenkongruenzen des zweiten Raumes und zugleich ein gewisses System von 2^{n-1} isotropen Kurvenkongruenzen des zweiten Raumes auf 2^{n-1} komplexe geodätische Normalenkongruenzen des ersten Raumes abbildet, so ist die Abbildung geodätisch. Die Fundamentaltensoren der beiden Räume lassen sich auf die klassische Form von Levi-Cività bringen.

Autoreferat (Zbl **51**, 125).

Teleman, C. Sur les espaces symétriques V_5. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 193–202. (Romanian. Russian and French summaries)

The author corrects one of the results of a previous paper [same Stud. **4** (1953), 503–526; MR **16**, 623] concerning the rotation group

$$X=X_{12}+2X_{34},\quad Y=\sqrt{3}X_{35}+X_{13}+X_{24},$$
$$Z=\sqrt{3}X_{15}-X_{14}+X_{23}.$$

This leads him to V_5 for which

$$ds^2=e^{-2mx^4+2\sqrt{3}mx^5}(dx^1-2mx^3dx^2)^2+e^{2mx^4+2\sqrt{3}mx^5}(dx^2)^2+e^{-4mx^4}(dx^3)^2+(dx^4)^2+(dx^5)^2.$$

The V_5 of E. Cartan [Bull. Soc. Math. France **55** (1927), 114–134, esp. p. 128] belongs to the case $m^2>0$; here the coordinates can be taken as the ratios of the coefficients of a definite quadratic form $a_{ij}\xi^i\xi^j$ of 3 variables, and the stability group is $a_{ij}'=p_i^r p_j^s a_{rs}$ (p_i^r real constants). Through two arbitrary points of the V_5 passes always one and only one geodesic. *D. J. Struik* (Cambridge, Mass.).

Vrănceanu, G. Sur les espaces V_n à groupe simplement transitif. Rev. Math. Phys. **2** (1954), 51–58 (1955).

In a Riemannian V_n with metric $ds^2=(ds^1)^2+\cdots+(ds^n)^2$, where $ds^h=\lambda_i^h dx^i$ are the arc differentials of n orthogonal congruences with constant rotation coefficients γ_{kl}^h, there exists a real simply transitive group of motions G_n, and conversely [Vrănceanu, Bul. Soc. Ştiinţ. Cluj **6** (1932), 429–445]. The structure constants of G_n are $c_{kl}^h=\gamma_{kl}^h-\gamma_{lk}^h$. The vector $c_s=c_{hs}^h$ is called the structure vector; if it is not zero, its direction is that of a geodesic congruence. It is shown that a necessary condition that the Ricci tensor $\gamma_{bd}=\gamma_{ja}^a\gamma_{bd}^j-\gamma_{af}^b\gamma_{fa}^d$ be positive definite is that G_n coincides with its derived group G_n', and that, if G_n is semi-simple the congruences with respect to which the tensor $c_{qs}=c_{qh}^k c_{sk}^h$ is reduced to canonical form are geodesic congruences. Another result is that if a non-integrable G_n coincides with its G_n' and its largest integrable group is G_2, then the G_2 is abelian; if a nonintegrable G_n has a G_3 as its largest integrable group, and the G_3 is of rank zero, then G_n coincides with its G_n'. *D. J. Struik.*

Vrănceanu, Gh. Sur les groupes de mouvements d'un espace de Riemann. Com. Acad. R. P. Române **1** (1951), 137–140. (Romanian. Russian and French summaries)

A theorem by I. P. Egorov [Dokl. Akad. Nauk SSSR (N.S.) **66** (1949), 793–796; MR **11**, 211] states that a Riemannian V_n which is not an Einstein space has a maximum group of automorphisms of $\frac{1}{2}n(n+1)+1$ parameters and that this maximum is reached. It is now shown that if the V_n is not conformally Euclidean this maximum is $\frac{1}{2}(n-1)(n-2)+3$ and that this maximum is also reached; moreover, if the Einstein V_n has no constant curvature, the maximum is $\frac{1}{2}(n-1)(n-2)+5$, reached for $n=4$ and $n=6$, but not reached for $n\geqq 7$.

D. J. Struik (Cambridge, Mass.).

Vrănceanu, Gh. Espaces à connexion affine constante localement euclidiens. Com. Acad. R. P. Române **1** (1951), 29–34. (Romanian. Russian and French summaries)

The equivalence in the large of an affine A_n of constant Γ_{jk}^i with an affine-euclidean E_n depends on the integration of the equations

$$\frac{\partial^2 u}{\partial x^j \partial x^k}+\Gamma_{jk}^r\frac{\partial u}{\partial x^r}=0,$$

$\Gamma_{kf}^i\Gamma_{jl}^f-\Gamma_{lf}^i\Gamma_{jk}^f=0$. This equivalence does not exist if solutions of the form $u=e^{\alpha_i x^i}$ exist (α_i constant). If solutions of the form $u=\alpha_i x^i$ exist, a sufficient condition for equivalence is established. The solutions are always integer functions, and equivalence in the large exists if both the solutions $u^i=u^i(x^1, x^2, \cdots, x^n)$ and their inverse $x^i=x^i(u^1, u^2, \cdots, u^n)$ are integer. The x^i satisfy the equations

$$\frac{\partial^2 x^i}{\partial u^j \partial u^k}=\Gamma_{rs}^i\frac{\partial x^r}{\partial u^j}\frac{\partial x^s}{\partial u^k}.$$

D. J. Struik (Cambridge, Mass.).

Vrănceanu, G. Sur la réduction à une forme canonique des équations des courbes auto-parallèles d'un espace A_2. Com. Acad. R. P. Române **2** (1952), 479–484. (Romanian. Russian and French summaries)

The differential equation of the geodesics of a space A_2 of symmetrical affine connection is

$$\frac{d^2y}{dx^2}=a\left(\frac{dy}{dx}\right)^3+b\left(\frac{dy}{dx}\right)^2+c\frac{dy}{dx}+d,$$

where a, b, c, d are functions of x and y. A point transformation can reduce two of these coefficients to zero. The cases $a=d=0$ and $a=c=0$ are singled out and an application is made to the geodesics of a Riemannian surface V_2. *D. J. Struik* (Cambridge, Mass.).

Vrănceanu, G. Réduction à une forme canonique d'une connexion linéaire de l'espace A_2. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **5** (1953), 481–484. (Romanian. Russian and French summaries)

The author [see the paper reviewed above] has established different canonical forms to which the connection coefficients of an affine A_2 can be reduced. In the present paper he considers the case that $\Gamma_{jk}^i=A_{jkl}^i x^l+A_{jk}^i$, where the A are constants. When Γ_{11}^1 and Γ_{11}^2 are linearly independent, then a linear transformation of variables can reduce these coefficients to the forms

a) $\Gamma_{11}^1=x^2$, $\Gamma_{11}^2=\pm x^1+ax^2$ (a constant);

b) $\Gamma_{11}^1=\varepsilon x^1$, $\Gamma_{11}^2=ax^2$ ($\varepsilon=\pm 1$, $a\neq\varepsilon$);

c) $\Gamma_{11}^1=\varepsilon x^1+bx^2$, $\Gamma_{11}^2=\varepsilon x^2$ ($\varepsilon=\pm 1$, $b=0, 1$).

There are also three cases when Γ_{11}^1 and Γ_{11}^2 are not linearly independent. When Γ_{11}^1 and Γ_{11}^2 are constant, the canonical expressions are a) $\Gamma_{11}^1=0$, $\Gamma_{11}^2=\pm 1$; b) $\Gamma_{11}^1=1$, $\Gamma_{11}^2=0$; c) $\Gamma_{11}^1=\Gamma_{11}^2=0$. *D. J. Struik.*

Vrănceanu, Gh. Espaces A_2 localement euclidiens. Com. Acad. R. P. Romîne **4** (1954), 335–340. (Romanian. Russian and French summaries)

The determination of all affine A_2 which are locally euclidean (torsion and curvature tensor zero) depends in the general case where the Γ_{jk}^i are not zero, on the solution of the equation (x, y are the coordinates of the A_2):

$$\frac{\partial^2\lambda}{\partial x\partial y}-\frac{1}{1+x+y}\left(e^\lambda\frac{\partial\lambda}{\partial x}+e^{-\lambda}\frac{\partial\lambda}{\partial y}\right)+\frac{1}{(1+x+y)^2}(e^\lambda-e^{-\lambda})=0.$$

Special cases, in which some Γ_{jk}^i vanish, are discussed. In the case that all Γ_{jk}^i except Γ_{11}^1, Γ_{22}^2 vanish the integration can be explicitly performed and Γ_{11}^1, Γ_{22}^2 can be takes as linear functions of x and y respectively.

D. J. Struik (Cambridge, Mass.).

Petrescu, Şt. Sur les espaces non holonomes A_n^m totalement géodésiques. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 51–57. (Romanian. Russian and French summaries)

In an earlier paper [Disquisit. Math. Phys. **4** (1945), 117–130; MR **8**, 604], the author has shown that a necessary condition that an A_n with a symmetrical affine connection allow non-holonomic A_n^m which are totally geodesic, depending on $\frac{1}{2}n(n+1)(n-m)$ arbitrary con-

stants, is that A_n be projectively plane. It is now shown that this condition is also sufficient (as a consequence of Bianchi's identity). The A_n^m can be considered as the intersections of $n-m$ A_n^{n-1} depending on $\frac{1}{2}n(n+1)$ arbitrary constants and totally geodesic in A_n.

D. J. Struik (Cambridge, Mass.).

Mocanu, P. Sur la classification des espaces A_3 à connexion affine constante, localement euclidiens. Com. Acad. R. P. Române **1** (1951), 239–243. (Romanian. Russian and French summaries)

An A_3 of constant affine connection, without curvature and torsion, is only equivalent in the large with euclidean affine space E_3 if the vector $\Gamma_k=\Gamma_{ik}{}^i$ vanishes. In this case the connection of the A_3 can be reduced to one of the forms $\Gamma_{22}{}^1=1$, $\Gamma_{23}{}^1=1$, $\Gamma_{22}{}^1=\Gamma_{33}{}^1=1$ if the matrix A has rank 1, or to the form $\Gamma_{33}{}^1=\Gamma_{13}{}^2=1$ if this rank is 2; here A is:

$$\left\|\begin{matrix}\Gamma_{11}{}^1 & \Gamma_{12}{}^1 & \cdots & \Gamma_{33}{}^1\\ \Gamma_{11}{}^2 & \Gamma_{12}{}^2 & \cdots & \Gamma_{33}{}^2\\ \Gamma_{11}{}^3 & \Gamma_{12}{}^3 & \cdots & \Gamma_{33}{}^3\end{matrix}\right\|.$$

D. J. Struik (Cambridge, Mass.).

Mocanu, P. Espaces à connexion affine constante, équivalents en grand avec l'espace euclidien. Com. Acad. R. P. Române **2** (1952), 389–395. (Romanian. Russian and French summaries)

The results of the paper reviewed above are extended to an A_n, which is equivalent in the large with a euclidean affine E_n if Γ_k vanishes. For such a space the commutative and associative algebra associated to the A_n and given by $x=x^1e_1+\cdots+x^ne_n$, with characteristic equation

$$|x^s\Gamma_{js}{}^i-\delta_j{}^i(\omega)|=\delta(\omega)=0,$$

is nilpotent. *D. J. Struik.*

Mutō, Yosio. On the curvature affinor of an affinely connected manifold A_n, $n\geqq 7$, admitting a group of affine motions G_r of order $r>n^2-2n$. Tensor (N.S.) **5** (1955), 39–53.

Necessary conditions are deduced for an A_n with a symmetric affine connection, admitting a group of affine motions G_r. For $r>n^2-2n$ and $n\geqq 7$ the conditions are

$$R_{\cdot\mu\nu\omega}^{\lambda}=A^{\lambda}B_{\mu\nu\omega}+C^{\lambda}D_{\mu\nu\omega}+\delta_\mu{}^\lambda(P_{\nu\omega}-P_{\omega\nu})+\delta_\nu{}^\lambda P_{\mu\omega}-\delta_\omega{}^\lambda P_{\mu\nu},$$

where $R_{\cdot\mu\nu\omega}^{\lambda}$ is the curvature tensor and A, C, D and P are certain other tensors. For $r>n^2-n$ and $n\geqq 5$ the conditions are

$$R_{\cdot\mu\nu\omega}^{\lambda}=A^{\lambda}B_{\mu\nu\omega}+\delta_\mu{}^\lambda(P_{\nu\omega}-P_{\omega\nu})+\delta_\nu{}^\lambda P_{\mu\omega}-\delta_\omega{}^\lambda P_{\mu\nu}.$$

The author promises more precise conditions in another paper. *J. A. Schouten* (Epe).

Wang, Hsien-Chung, and Yano, Kentaro. A class of affinely connected spaces. Trans. Amer. Math. Soc. **80** (1955), 72–92.

The question whether a given space $\mathfrak{A}_n$ with a symmetric affine connexion admits a G-dimensional group of affine motions is dealt with by several authors. [For a review of results see J. A. Schouten, Ricci-calculus, 2nd ed., Springer, Berlin, 1954, p. 348; MR **16**, 521]. In this paper it is proved first (Th. I) that if G is a closed and connected subgroup of P_n (special linear group) and $G\geqq n^2-2n+4$, then G is conjugate to P_n or to one of four well-defined subgroups of P_n. Then (Th. II) all the possible curvature tensors of $\mathfrak{A}_n$ as well as the isotropic subgroups of G are determined. For the case that G is transitive all the simply connected $\mathfrak{A}_n$'s are exhibited. They fall into three individual cases and two classes depending on a non-negative constant. Among them, there are three affinely flat spaces, two of which are homeomorphic with the product of a line and an $(n-1)$-sphere, $n>2$. As for the non-flat cases, they are all homeomorphic with the euclidean space. For the transitive case, all the isotropic subgroups given in Theorem II are realized; some of them can moreover by realized over nonequivalent affinely connected spaces. *J. A. Schouten* (Epe).

Nalli, Pia. Equazioni indipendenti dalla scelta delle variabili e caratterizzazione di varietà metriche. Boll. Un. Mat. Ital. (3) **10** (1955), 135–146.

From the theory of representations we know that the only sets of variables that transform linearly and homogeneously with the transformations of the homogeneous affine group are the components of tensor densities. Hence, if the author claims to have constructed other sets with this property, that are not components of tensor densities, there must be some misunderstanding. In fact the set of n^3-n variables deduced from a tensor $P_{\mu\lambda}^{\cdot\cdot\cdot\varkappa}$, viz. $P_{\mu\lambda}^{\cdot\cdot\cdot\mu}-\frac{1}{2}P_{\mu\mu}^{\cdot\cdot\cdot\mu}$; $P_{\lambda\mu}^{\cdot\cdot\cdot\mu}-\frac{1}{2}P_{\mu\mu}^{\cdot\cdot\cdot\mu}$; $\lambda\neq\mu$; $P_{\mu\lambda}^{\cdot\cdot\cdot\varkappa}$; $\varkappa\neq\lambda$, μ, that the author gives us as an example, is really only a set of components for the combination of the two tensors

$$\overset{1}{P}_\lambda-\overset{2}{P}_\lambda,\quad P_{\mu\lambda}^{\cdot\cdot\cdot\varkappa}-(1-n^2)^{-1}A_\mu{}^\varkappa\overset{3}{P}_\lambda+n(1-n^2)^{-1}A_\mu{}^\varkappa\overset{1}{P}_\lambda+$$

$$n(1-n^2)^{-1}A_\lambda{}^\varkappa\overset{2}{P}_\mu-(1-n^2)^{-1}A_\lambda{}^\varkappa\overset{1}{P}_\mu,$$

with n and n^3-2n components and it can be derived from the ordinary set of components written out here by a linear homogeneous transformation. The vanishing of the author's set is exactly equivalent with the vanishing of these two tensors and its geometric meaning is that $P_{\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ can be written in the form $p_\lambda A_\mu{}^\varkappa+p_\mu A_\lambda{}^\varkappa$. Of course, it is always possible to take some linear combinations of components of a tensor or a set of tensors, and these combinations will transform in a queer (but linear homogeneous) way (for instance $v^1+\frac{1}{2}v^2$, v^2, v^3 for a vector in E_3), but this process can never bring anything new. Taking this into account, the author's questions about physical facts that could be expressed not by tensor equations but only by equations containing these new sets, may be disregarded. *J. A. Schouten* (Epe).

Mihăileanu, N. N. Objets géométriques associés aux espaces à connexion projective P_2. Com. Acad. R. P. Române **1** (1951), 165–170. (Romanian. Russian and French summaries)

The connection defined by the $\Gamma_{kl}{}^r$ subjected to the transformation

$$\frac{\partial^2x'^i}{\partial x^k\partial x^l}=\Gamma_{rs}{}'{}^i\frac{dx'^r}{dx^k}\frac{dx'^s}{dx^l}-\Gamma_{kl}{}^r\frac{\partial x'^i}{\partial x^r}-\frac{\partial x'^i}{\partial x^k}a_l-\frac{\partial x'^i}{\partial x^l}a_k,$$

$$\frac{\partial a_k}{\partial x^l}=\Gamma_{rs}{}'{}^0\frac{\partial x'^r}{\partial x^k}\frac{\partial x'^s}{\partial x^l}-\Gamma_{kl}{}^0-a_ka_l-\Gamma_{kl}{}^ra_r;\ i,j,k,l,\cdots=1,2,$$

is subjected to the transformation $x'^1=x'^1(x^1)$, $x'^2=x'^2(x^2)$, $dx'^0=dx^0+a_1dx^1+a_2dx^2$. Then

$$\Gamma_{22}{}^1,\ T_{12}{}^1=\Gamma_{12}{}^1-\Gamma_{21}{}^1,\ \frac{\partial\Gamma_{12}{}^2}{\partial x^2}+\Gamma_{12}{}^1\Gamma_{12}{}^2-\Gamma_{12}{}^0$$

and three others obtained by $1\leftrightarrow 2$ are invariants, and

$$2\Gamma_{12}{}^2-\Gamma_{11}{}^1,\ \frac{\partial\Gamma_{11}{}^1}{\partial x^1}+\frac{1}{2}(\Gamma_{11}{}^1)^2-2\Gamma_{11}{}^0+2\Gamma_{12}{}^1\Gamma_{11}{}^2$$

and their analogues are geometrical objects, of which the second one transforms like a tensor plus an additional Schwartzian of x'^1 with respect to x^1. *D. J. Struik.*

Haimovici, Ad. Espaces à métrique angulaire. Com. Acad. R. P. Române **1** (1951), 157–163. (Romanian. Russian and French summaries)

Given is a two-dimensional S_2 with connection given by $dX^i+\Gamma_{kl}{}^i X^k dx^l = X^i d\varphi$, of which the angle between two vectors is given by a function $V(x^i, X^i, Y^i)$ which is additive, invariant under parallel displacement, homogeneous of degree zero in the two vectors and invariant under a coordinate transformation. If $R_{121}{}^1 = R_{122}{}^2$, $R_{121}{}^2 = R_{122}{}^1 = 0$, then

$$V = f\left(\frac{a_1X^2+a_2X^1}{a_3X^2+a_4X^1}\right) - f\left(\frac{a_1Y^2+a_2Y^1}{a_3Y^2+a_4Y^1}\right)$$

and absolute parallelism exists. If these relations are not fulfilled, the search leads to the types

$$V = \frac{K}{R_{122}{}^1(\alpha_2-\alpha_1)} \log \frac{X^2Y^2-\alpha_1X^1Y^2-\alpha_2X^2Y^1+\alpha_1\alpha_2X^1Y^1}{X^2Y^2-\alpha_1X^2Y^1-\alpha_1X^1Y^2+\alpha_1\alpha_2X^1Y^1},$$

$$(R_{122}{}^1\alpha^2+(R_{121}{}^1-R_{122}{}^2)\alpha-R_{121}{}^2=0,$$

$$\cos V = \frac{aX^2Y^2+\frac{1}{2}b(X^1Y^2+X^2Y^1)+cX^1Y^1}{\{a(X^2)^2+bX^1X^2+c(X^2)^2\}^{\frac{1}{2}}\{a(Y^2)^2+bY^1Y^2+c(Y^1)^2\}^{\frac{1}{2}}}.$$

D. J. Struik (Cambridge, Mass.).

Haimovici, Adolf. Espaces à métrique angulaire. II. Acad. R. P. Române. Fil. Iaşi. Stud. Cerc. Şti. **2** (1951), 66–82. (Romanian. Russian and French summaries)

[Pour la première partie voir l'analyse ci-dessus.] Soient S un espace à connexion affine, à deux dimensions, x^i les coordonnés d'un point de cet espace, X^i et Y^i les composantes de deux directions. Le transport parallèle dans cet espace est défini par $dX^i+\Gamma_{kl}{}^i X^k dx^l = wX^i$, $w=w_k dx^k$. On définit comme angle dans S une fonction $V(x^i; X^i; Y^i)$ qui satisfait aux axiomes suivants: a) elle est indépendante du facteur arbitraire qui entre dans les composantes X^i d'une part, et Y^i d'autre part; b) elle est invariante aux transformations de coordonnées dans S; c) elle se conserve par le transport parallèle des directions X^i et Y^i. Les équations qui définissent cette fonction et les espaces qui admettent de telles fonctions s'obtiennent en exprimant ces conditions mêmes. On traite quelques cas spéciaux, y compris le cas des espaces à parallélisme absolu. Si S est un espace à n dimensions qui admet un angle satisfaisant aux axiomes a), b), c) de plus haut, la fonction V doit satisfaire aux équations

$$X^i\partial V/\partial X^i = 0, \quad Y^i\partial V/\partial Y^i = 0,$$

$$\partial V/\partial x^k - \Gamma_{kl}{}^i X^k \partial V/\partial X^i - \Gamma_{kl}{}^i Y^k \partial V/\partial Y^i = 0.$$

La condition pour qu'un tel angle existe est qu'il y ait un entier p, tel que les équations du système

$$(p) \qquad R_{ijk,n_1\cdots n_p}{}^h(X^k\partial V/\partial X^k + Y^k\partial V/\partial Y^k) = 0$$

soient toutes linéairement dépendantes des équations précédentes (q), $q=1, 2, \cdots, p-1$, et que, si m est le nombre des équations linéairement indépendantes, on ait $m \leqq 2n-3$. La métrique angulaire est donnée par une fonction de $2n-m-2$ variables indépendantes. Cas particuliers: α) Un espace à courbure et à torsion nulle; β) on impose encore l'axiome d): la mesure de l'angle de deux directions dans S est égale à la mesure de l'angle des mêmes directions considérées dans chaque sous-espace à deux dimensions qui les contient. *Autoreferat* (Zbl **45**, 253).

Laugwitz, Detlef. Zur geometrischen Begründung der Parallelverschiebung in Finslerschen Räumen. Arch. Math. **6** (1955), 448–453.

In recent years several distinct covariant derivatives have been defined for Finsler spaces. In the present paper the author adapts the method of the osculating Riemannian space due to O. Varga [Monatsh. Math. Phys. **50** (1941), 165–175; MR **5**, 218] to obtain these derivatives from a unified point of view. Let C: $x^i = x^i(s)$ be a curve of a Finsler space F_n, $\zeta^i(s)$ a vector field defined along C. This determines a "Riemannian" metric tensor $g_{ij} = g_{ij}(x^i(s), \zeta^i(s))$ along C, where $g_{ij}(x^i, x'^i)$ is the metric tensor of F_n. The usual Christoffel symbols are formed with respect to these special g_{ij}. When two distinct sets of conditions are imposed on C and the on vector field $\zeta^i(s)$, the distinct covariant derivatives of the reviewer [Math. Z. **54** (1951), 115–128; MR **13**, 159] and of W. Barthel [Arch. Math. **4** (1953), 346–354, 355–365; MR **15**, 556] are obtained. The paper concludes with a discussion of the reviewer's geometrical construction of the covariant derivative. *H. Rund* (Toronto, Ont.).

Complex Manifolds

Kreyszig, E. Stetige, analytische und eigentliche Modifikationen komplexer Mannigfaltigkeiten. Wiss. Z. Ernst Moritz Arndt-Univ. Greifswald. Math.-Nat. Reihe **4** (1954/1955), 457–463.

Let M and M' be complex manifolds of the same dimension, and $N \subset M$, $N' \subset M'$ be closed subsets. M' is called a modification of M at N [Behnke and Stein, Math. Ann. **124** (1951), 1–16; MR **13**, 644] if there is a one-to-one analytic mapping T: $M'-N' \to M-N$ (onto) such that to any open neighborhood U of N and point $P' \in N'$, there corresponds a neighborhood U' of P' such that $T[U' \cap (M'-N')] \subset U-N$. The modification is defined to be continuous, analytic, or proper, according as T can be extended to a map T^*: $M' \to M$ which is continuous, analytic, or proper, respectively (a continuous map is proper if the pre-image of every compact set is compact). The author proves the following results: every continuous modification at an analytic subvariety N is an analytic modification; for proper analytic modifications, the rings of holomorphic functions and fields of meromorphic functions of M and M' are isomorphic under T^*; moreover if N and N' are irreducible analytic subvarieties of the same dimension in a proper analytic modification, then the modification is trivial in the sense that M and M' are analytically homeomorphic under T^*. *R. C. Gunning.*

Bochner, S., and Gunning, R. C. Existence of functionally independent automorphic functions. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 746–752.

Sur une variété analytique D à n dimensions complexes, ayant un groupe dénombrable Γ d'automorphismes analytiques, avec ou sans point fixe, on dira [cf. S. Bochner, J. Indian Math. Soc. (N.S.) **11** (1947), 1–21; MR **9**, 423] que les fonctions $\eta_T(z)$, méromorphes sur D, $T \in \Gamma$, constituent un ensemble de facteurs d'automorphie si l'on a

$$\eta_{ST}(z) = \eta_S(Tz)\eta_T(z).$$

Un tel ensemble est dit complet si pour chaque $\alpha \in \Gamma$, il existe a) une fonction $h_\alpha(z) \neq 0$, holomorphe, avec $h_\alpha[\eta_{\alpha,T}]^{-1}$ holomorphes; b) un sous-groupe Γ_α de Γ avec $\eta_{\alpha,T} \equiv 1$ pour $T \in \Gamma_\alpha$, la série $\sum_T h_\alpha[\eta_{\alpha,T}]^{-1}$, T parcourant

la classe de restes Λ_α de Γ_α dans Γ, étant convergente uniformément sur tout compact de D vers une fonction $f(z)\neq 0$; c) un sous ensemble $M\subset D$ (de dimension au plus $2n-1$, et réunion dénombrable de compacts) tel que pour tout $z_0 \in D-M$, chaque $\eta_{\alpha,T}$ est holomorphe dans un voisinage de z_0, l'ensemble $\sum_{\alpha,T} E[\eta_{\alpha,T}(z)=\eta_{\alpha,T}(z_0)]$ étant de dimension zéro dans un voisinage de z_0. On établit alors: si (D, Γ) admettent un système complet de facteurs d'automorphie, il existe au moins n fonctions automorphes indépendantes. Application de ce résultat est faite aux fonctions multipériodiques, puis au cas où D est un domaine borné de l'espace affine, Γ étant proprement discontinu, enfin au cas où D est l'espace des matrices Z carrées, d'ordre p, avec $-i(Z-\bar{Z})>0$, Γ étant le groupe modulaire de degré p: $Z\to(AZ+B)(CZ+D)^{-1}$; les $\eta_T=\det(CZ+D)^{p+2}$ forment alors un système complet de facteurs d'automorphie. *P. Lelong* (Paris).

Frölicher, Alfred. Relations between the cohomology groups of Dolbeault and topological invariants. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 641–644.

The author considers the cohomology groups $H^p(M, \Omega^q)$ of a complex-analytic manifold M of complex dimension m with coefficients in the sheaf Ω^q of germs of holomorphic differential forms of degree q. It is shown that the groups $H^p(M, \Omega^q)$ form the term E_1 of a spectral sequence of which E_∞ is the associated C-module of the conveniently filtered cohomology group of the complex-valued forms on M of class C^∞. The sequence is stationary after a finite number of steps. With the aid of this spectral sequence it is shown that the Euler characteristic χ of M equals $\sum_{p,q=0}^{n}(-1)^{p+q}b_{p,q}$, where $b_{p,q}$ is the dimension of $H^p(M, \Omega^q)$. Moreover, the rth Betti number of M cannot exceed $\sum_{p+q=r} b_{p,q}$ $(0\leq r\leq 2m)$. *P. A. Smith.*

Nakano, Shigeo. On complex analytic vector bundles. J. Math. Soc. Japan **7** (1955), 1–12.

The main result here is to be found in Theorem 4 of this article which can be briefly stated as follows: Let $\mathcal{F}$ be a complex analytic bundle of r-dimensional complex vector spaces over a non-singular algebraic variety V and let $\mathcal{B}$ be the complex line bundle over V associated with a general hyperplane section. Then for some large positive integer m there exists a regular complex analytic mapping Φ (i.e., the local coordinates of $\Phi(p)$ are complex analytic functions of the local coordinates of p) of V into a Grassmann variety $H(r, M)$ (variety of r-dimensional subspaces of C^M), which serves as the base space for a universal bundle $\mathcal{R}$ of r-dimensional vector spaces, such that $\Phi^{-1}(\mathcal{R})$ is defined by the transition functions

$$ {}^t h_{\alpha\beta}{}^{-1}: U_\alpha \cap U_\beta \to \mathrm{GL}(r, C), $$

where $h_{\alpha\beta}$ are the transition functions for $\mathcal{B}^m\mathcal{F}$, $\{U_\alpha\}$ is an indexed covering of V, $\mathrm{GL}(r, C)$ is the general complex linear group on r variables, and ${}^t h_{\alpha\beta}{}^{-1}(p)$ is the transposed inverse of $h_{\alpha\beta}(p)$. From this the author is able to write an expression for the characteristic polynomial of $\mathcal{F}$. This result is a natural partial generalization of Chern's classifying theorem [Ann. of Math. (2) **47** (1946), 85–121; MR **7**, 470]. However, nothing is said as to when two such maps Φ and Φ' of V into $H(r, M)$ induce complex analytically equivalent bundles to compare with Chern's result that continuous equivalence classes of bundles correspond to homotopy classes of maps; and no issue is made of whether or not $\Phi(V)$ is an analytic subvariety of $H(r, M)$.

A good deal of motivation for the article seems to lie in Chern's result [Amer. J. Math. **75** (1953), 565–597; MR **15**, 154], that the characteristic classes of a non-singular algebraic variety are dual to algebraic cycles, and in Kodaira and Spencer's result that every complex line bundle on an algebraic variety comes from a divisor [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 872–877; MR **16**, 75]. The proof rests on generalizations of Kodaira's differential-geometric method [ibid. **39** (1953), 1268–1273; MR **16**, 618] and Kodaira's method of imbedding a variety in projective space [Ann. of Math. (2) **60** (1954), 28–48; MR **16**, 952]. As the author remarks, the same result has been obtained by J. P. Serre by different methods. More specifically, Theorem 2 of this article, from which the principal results follow easily, is an easy consequence of a theorem on analytic coherent sheaves over an algebraic variety [Séminaire Bourbaki, Faisceaux analytiques, by J. P. Serre, March 1954]. It is unfortunate that the article is so condensed as to make apparently impossible a more detailed exposition of certain points. *W. L. Baily.*

Thimm, Walter. Über die Nullstellenmengen von Polynomidealen über dem Potenzreihenring. J. Reine Angew. Math. **193** (1954), 183–208.

Dans l'espace des $m+k$ variables complexes $C^m(x_j)\times C^k(z_s)=C^{m+k}$, x_j, z_s complexes, $x=(x_j)$, $z=(z_s)$, on considère essentiellement des domaines se projetant sur $C^m(x)$ dans un voisinage de son origine $x=0$, qui sera désignée par $\mathfrak{O}$, des germes d'ensembles analytiques en des points projetés en $\mathfrak{O}$ et des idéaux dans l'anneau des séries en x_j convergentes dans un voisinage de $\mathfrak{O}$; $C^k(z)$ est supposé fermé par un groupe de transformations linéaires de manière que toute variété analytique compacte y est algébrique. M étant une variété analytique dans un voisinage U_M de P, si l'idéal de M dans l'anneau des séries de centre P a une base cohérente dans U_M, on dira que $(M)=M\cap U_M$ est un domaine analytique de centre P; ce domaine sera dit premier si (M) est un germe premier en P; pour $P=(\mathfrak{O}, z)$, M sera dit germe $\mathfrak{O}$-premier et (M) domaine $\mathfrak{O}$-premier si l'on a:

$$ U_M=[x\in X,\ z\in Z],\quad (\delta)=(M)\cap[x=\mathfrak{O},\ z\in Z], $$
$$ [\delta]=M\cap[x=\mathfrak{O},\ z\in \bar{Z}], $$

chaque point de $[\delta]$ se laissant joindre à $[\mathfrak{O}, z]$ par une courbe dans (δ) et si $\delta_1\cap(\delta)$ n'est pas vide, δ_1 étant un germe premier quelconque de δ en un point frontière $[\mathfrak{O}, z_1]$ de (δ). Un domaine $\mathfrak{O}$-analytique est constitué par une réunion dénombrable de domaines $\mathfrak{O}$-premiers. Il sera dit $\mathfrak{O}$-connexe s'il en est une réunion finie, $\mathfrak{O}$-compact s'il existe un voisinage U de $x=0$ tel que point tout compact $K\subset U$, l'ensemble des points de l'espace des germes $\overline{\mathfrak{M}}$ au dessus de $[K\times C^k(z)]$ soit compact. On établit alors que si $\mathfrak{M}$ est un domaine analytique $\mathfrak{O}$-compact, il existe un voisinage U de $x=0$, tel que la projection de $(U\times C^k(z))\cap\mathfrak{M}$ sur $C^m(x)$ soit une variété analytique complexe dans U. [Note: ce résultat, ainsi que ceux qui suivent, sont établis indépendamment des résultats donnés par R. Remmert et K. Stein, Math. Ann. **126** (1953), 263–306; MR **15**, 615.] $\mathfrak{M}$ étant $\mathfrak{O}$-compact et $\mathfrak{O}$-connexe, on considère sa dimension complexe ϱ et sa hauteur (Stufe) σ définie comme la dimension complexe de la projection d'un germe premier de $\mathfrak{M}$ sur $C^m(x)$. On établit que si la projection de $\mathfrak{M}$ sur $C^k(z)$ se compose d'un nombre fini de points, il existe un idéal premier dans l'anneau des fonctions holomorphes à l'origine des (z_k), s'annulant sur $\mathfrak{M}$ et de dimension ϱ, de hauteur σ, idéal qui est unique; la variété de l'idéal ne contient en dehors de $\mathfrak{M}$ aucun domaine analytique $\mathfrak{O}$-connexe et $\mathfrak{O}$-compact, de dim ϱ,

de hauteur σ. Le cas général en résulte: à un idéal premier Γ, de dim ϱ, de hauteur σ correspond un seul domaine $\mathfrak{D}$-compact, $\mathfrak{D}$-connexe, $\mathfrak{M}(\Gamma)$, de dim ϱ, de hauteur σ sur lequel s'annule Γ, les autres domaines analytiques $\mathfrak{D}$-compacts, $\mathfrak{D}$-connexes formés avec des zéros de Γ ont une hauteur $\sigma' \leqq \sigma - 1$. *P. Lelong* (Paris).

Matsushima, Yozô. Un théorème sur les espaces homogènes complexes. C. R. Acad. Sci. Paris **241** (1955), 785–787.

The author proves the following theorem. Let $V=G/H$ be a complex homogeneous space (G connected, H compact and containing no subgroups other than (e) which are invariant in G). If the connected linear isotropy group is irreducible then V admits a symmetric Hermitian structure. *W. Ambrose* (Cambridge, Mass.).

See also: Bochner, p. 250; Mitchell, p. 253; Chevalley, p. 342; Yano, p. 404.

Algebraic Geometry

Teixidor, J. On a theorem of Lüroth. Gac. Mat., Madrid (1) **7** (1955), 79–83. (Spanish)

From the well-known proof of the Lüroth's theorem one obtains a computation of a proper parametric representation of a rational curve. The author gives a computation in which the polynomials involved are of lower degree. *E. Lluis* (Mexico, D.F.).

Lievens, Joseph. Sur une transformation birationnelle de Jonquières particulière. Mathesis **64** (1955), 249–253.

Vaccaro, Giuseppe. Sulle curve apolari. Giorn. Mat. Battaglini (5) **2(82)** (1954), 349–358.

The author studies the geometric meaning of the apolarity relation between a plane (point) curve C^n of order n and a line curve Γ^n, for particular degenerate curves C^n and for curves C^n with particular singularities. The results are used to describe the construction of linear systems of curves apolar to a given line curve Γ^n. *G. B. Huff* (Athens, Ga.).

Manara, Carlo Felice. Questioni di esistenza di curve algebriche piane con caratteri assegnati. Rend. Sem. Mat. Fis. Milano **24** (1952–53), 66–77 (1954).

The author discusses certain questions about the existence of plane algebraic curves with prescribed Plückerian characters. *G. B. Huff* (Athens, Ga.).

Brusotti, Luigi. Un teorema sui fasci reali di curve algebriche a curva reale generica massimale. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 240–242.

This note is a supplement to an earlier paper [Rend. Mat. e Appl. (5) **14** (1954), 239–251; MR **16**, 740], and settles a question of existence raised there. Specifically, it is shown that there exist real pencils Φ_0 (of genus zero) of curves of genus $p=2q$ which sweep out a real algebraic surface whose real part consists of $q+1$ components each of which is homeomorphic to a torus. *H. T. Muhly*.

Benedicty, Mario. Quelques considérations sur les genres d'une courbe algébrique par rapport à une relation d'équivalence généralisée. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 829–836.

Etant donné un corps R de fonctions algébriques d'une variable, on définit le genre de R comme étant l'entier

$$g = 1 + \sup_{\mathfrak{a}} (d(\mathfrak{a}) - r(\mathfrak{a})) \tag{1}$$

où $\mathfrak{a}$ parcourt l'ensemble E de tous les diviseurs de R, et où $d(\mathfrak{a})$ et $r(\mathfrak{a})$ désignent respectivement le degré de $\mathfrak{a}$ et la dimension de l'espace vectoriel $L(\mathfrak{a})$ des fonctions x telles que $(x)+\mathfrak{a} \geqq 0$. Si l'on se donne, en plus, un sous anneau $\mathfrak{o}$ de R admettant R pour corps des fractions, qu'on remplace $L(\mathfrak{a})$ par $\mathfrak{o} \cap L(\mathfrak{a})$, et qu'on fasse seulement parcourir à $\mathfrak{a}$ certaines parties de E (liées à $\mathfrak{o}$), les formules analogues à (1) définissent divers „genres" de R, qui sont entiers ou infinis. L'auteur établit quelques relations entre ceux ci. Ces relations ont des formes distinctes suivant que $\mathfrak{o}$ est semi-local ou non. *P. Samuel.*

Benedicty, Mario. Sull'equivalenza tra matrici normali di Severi. Ann. Mat. Pura Appl. (4) **38** (1955), 51–76.

L'Autore si occupa del seguente problema, da lui posto [Rend. Mat. e Appl. (5) **12** (1953), 332–339; MR **15**, 821]: determinare se ed in quali casi gli interi caratteristici p, δ_1, δ_2, ϱ ed i divisori elementari delle matrici normali di Severi [Funzioni quasi abeliane, Pont. Acad. Sci. Scripta Varia, v. 4, 1947; MR **9**, 578] equivalenti ad una prefissata matrice ω dello stesso tipo possan essere diversi da quelli della ω; intendendo che due matrici simili ω, ω^* siano equivalenti se esistono una matrice complessa non degenere β ed una matrice intera unimodulare B tali che $\omega^* = \beta\omega B$. Premesse alcune proprietà della relazione di equivalenza che conducono a delle condizioni necessarie perché due matrici normali di Severi possano essere equivalenti, l'Autore risolve completamente il problema per $p=\varrho$ e $p=\varrho+1$; per $p>\varrho+1$ si limita a matrici che soddisfano certe condizioni di genericità. I risultati principali che l'Autore ottiene sono già stati esposti in una sua precedente Nota [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 716–720; MR **16**, 666]. *D. Gallarati* (Genova).

Benedicty, Mario. Sulla definizione di gruppo unimodulare ristretto. Rend. Mat. e Appl. (5) **14** (1955), 368–381.

Se Δ è una matrice diagonale di ordine p i cui elementi principali $d_1, d_2, \cdots, d_p$ siano interi positivi tali che $d_1=1$ e d_{h-1} divida d_h ($h=2, 3, \cdots, p$) l'insieme GR[Δ] delle matrici intere unimodulari Γ d'ordine $2p$ tali che:

$$\Gamma M \Gamma_{-1} = M, \quad M = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

costituisce il gruppo modulare ristretto relativo al livello Δ. La sua considerazione sorge spontanea nello studio dell'equivalenza tra matrici normali di Riemann [cf. F. Conforto, Funzioni abeliane modulari, vol. I, Docet, Roma, 1952; MR **14**, 859]. L'Autore, nella Memoria recensito sopra, ha definito il gruppo unimodulare ristretto, che è l'estensione al caso quasi abeliano del GR[Δ] della teoria della matrici di Riemann. Nella Nota presente, l'Autore giustifica l'introduzione di tale gruppo, dimostrando tra l'altro una proposizione che sostituisce nella teoria delle matrici normali di Severi un fondamentale teorema dovuto ad F. Conforto, della teoria delle matrici normali di Riemann [Conforto, loc. cit., p. 122]. *D. Gallarati* (Genova).

Spampinato, Nicolò. La 1ª rappresentazione complessa della ipersuperficie generale dell' S_r quadripotenziale. Ricerche Mat. **3** (1954), 172–188.

A hypersurface in projective S_r over an algebra of

hypercomplex numbers of order m has a representation in complex projective S_N, $N=m(r+1)-1$. The varieties arising in this way in the case of the bi-dual and tri-potential algebras have been studied by the author [Ricerche Mat. 3 (1954), 13–30; Rend. Accad. Sci. Fis. Mat. Napoli (4) 20 (1954), 58–65; MR 16, 510, 614]. For the quadri-potential algebra, this representation is a V_{4r-1} in S_{4r+3} defined by four equations. In the present paper, this variety is studied independently of its origin. *G. B. Huff* (Athens, Ga.).

Spampinato, Nicolò. Un modello proiettivo dell'ente algebrico ∞^{r-1} dell' S_r complesso ottenuto con l'algebra dei numeri $(r+1)$-duali. Giorn. Mat. Battaglini (5) 2(82) (1954), 359–377.

Beginning with the observation that the r-fold bidual algebra over the complex numbers has a $(r+1)$-dual subalgebra, the author constructs and studies a projective model W_{d-1} in complex S_R $(R=2r(r+1)-1)$, of a given algebraic hypersurface in complex projective S_r. Explicit formulas are given for the representation in S_R of birational transformations in the $(r+1)$-dual algebra obtained by extending birational transformations of complex S_r. The behavior of this rational transformation with respect to varieties of type W_{d-1} is studied. *G. B. Huff.*

Hall, R. On algebraic varieties which possess finite continuous commutative groups of birational self-transformations. J. London Math. Soc. 30 (1955), 507–511.

Algebraic geometry splits, so-to-speak, into classical and modern algebraic geometries (while the latter splits further into transcendental and abstract algebraic geometries.) This paper belongs to classical algebraic geometry, but, in the terminology of modern algebraic geometry, the result can be explained in the following way: Let V be an algebraic variety over the complex field, and let G be an algebraic group acting on V. Let F be the subvariety of the product $G\times V\times V$, which represents the operations of elements of G on V. Let P be a general point of V, and let T_P be the projection on V of the intersection $F\cap G\times P\times V$. Then, by Severi's theorem, T_P decomposes birationally into a product of a projective space S and an Abelian variety A. The author claims that V decomposes birationally to the product of S and a carrier variety of a fibre system, the fibre being the Abelian variety A. *J. Igusa* (Baltimore, Md.).

See also: Amitsur, p. 340; Stubban, p. 398; Lepskiĭ, p. 399; Mahler, p. 402.

NUMERICAL ANALYSIS

Orloff, Constantin. Application pratique de la théorie des spectres mathématiques de Michel Petrovitch au calcul numérique. Rev. Sci. 91 (1953), 243–247.

Exposition of author's developments [cf., e.g., Bull. Soc. Math. Phys. Serbie 5 (1953), no. 1–2, 17–30; MR 15, 472]. *John Todd* (Washington, D.C.).

Orloff, Constantin. Spectre mathématique des racines d'une équation algébrique. Bull. Soc. Math. Phys. Serbie 6 (1954), 56–62. (Serbo-Croatian summary)

Suppose all real zeros x of a polynomial $p(x)$ satisfy $a\leq x\leq b$, where a, b are integral. From the values $p(a+i\cdot 10^{-j})$, $j=0, 1, 2, \cdots$, $i=0, 1, \cdots, (b-a)10^j$, a symbol, in the form of an infinite decimal, can be constructed from which the real zeros, can, in principle, be read off, to any accuracy. *John Todd* (Washinton, D.C.).

Sarymsakov, T. A. On an approximate method of computing the real roots of polynomials belonging to sequences of polynomials with a regular distribution of roots. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 413–416. (Russian)

A sequence $\{P_n(x)\}$ of polynomials is said to have regular distribution of roots on (a, b) if all roots are in (a, b) and if $\lim n^{-1}N_n(-\infty, x)=\psi(x)$ exists, where $N_n(c, d)$ is the number of roots of $P_n(x)$ on $[c, d]$. The case where $\psi(x)$ is differentiable, $\psi'(x)=\phi(x)$, has been studied previously by the author [Trudy Uzbek. Filial. Akad. Nauk SSSR (4) 1 (1941); C.R. (Dokl.) Acad. Sci. URSS (N.S.) 24 (1939), 322–324; MR 2, 51]. Hermite and Legendre polynomials enjoy this property. A necessary and sufficient condition that $\psi(x)$ be the same for two sequences $\{\sum_k p_{kn}x^k\}$, $\{\sum_k q_{kn}x^k\}$ is stated to be $\lim_n p_{k,n}/q_{k,n}=1$. *J. L. Brenner* (Pullman, Wash.).

Hildebrand, F. B. Note on S. N. Lin's method of factoring polynomials. J. Math. Phys. 32 (1953), 164–170.

S. N. Lin [same J. 20 (1941), 231–242; MR 3, 153] gave a method for the computation of the roots of algebraic equations by factorization. Here is considered the special case in which a single quadratic factor is to be extracted, and a precise criterion for asymptotic stability of the iteration is obtained. For a quartic the result is geometrically interpreted by means of Cassinian ovals. Also, certain features of the method are discussed. *E. Frank.*

★ **Matsukura, Kameo. On a solution of equations containing general polynomials with numerical coefficients and certain transcendental functions by expanding their roots into infinite series.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 295–298. Science Council of Japan, Tokyo, 1953.

This paper describes the Lagrange method for the solution of equations $y=f(x)$ (with derivatives of all orders) by means of the Taylor series expansion of the inverse function $x=g(y)$ [cf., e.g. Fr. A. Willers, Practical analysis, Dover, New York, 1948; MR 10, 404]. *E. Frank.*

Müller, R. Eine rationale Näherung für $\sqrt{x^2+y^2}$. Z. Angew. Math. Mech. 35 (1955), 437–439.

Greenstadt, J. A method for finding roots of arbitrary matrices. Math. Tables Aids Comput. 9 (1955), 47–52.

The method proposed is that of triangularizing a matrix by a unitary similarity, $A\rightarrow S^*AS$ with $S^*S=1_n$, where S is obtained as a product of unitary matrices describing transformations in coordinate planes and chosen so as to annihilate a single element below the diagonal. An example of order three shows that the sum of squares of the absolute values of the sub-diagonal elements may increase at any particular step, yet experimental evidence is cited in support of the contention that the process converges to a triangular form and indeed with a number N of iterations ($\frac{1}{2}n(n-1)N$ of separate annihilations of elements) which varies about linearly with n. The experiments made are not described with sufficient precision for their significance to be evaluated (the matrices have

elements "picked at random" and convergence is "with floating point, to 10^{-30} for subdiagonal elements"). The number of iterations is said to increase considerably when roots are equal and the roots are said not to be accurate but an explanation of this inaccuracy is unconvincing. *W. Givens* (Knoxville, Tenn.).

Langefors, B. Approximate solution of simultaneous equations by means of transformation of variables. Applications to aeronautical problems. Svenska Aeroplan A. B. Tech. Note No. 7 (1953), 26 pp.

Bei manchen mechanischen oder elektrischen Aufgaben treten lineare Gleichungssysteme der Art auf, daß zwischen $2n$ Variablen $x_1, \cdots, x_n, y_1, \cdots, y_n$, die zu Spalten x, y zusammengefaßt werden mögen, Relationen der Gestalt $y=Vx$ bestehen, wo V eine Diagonalmatrix ist (statt Zahlen können allgemeiner in der Diagonale von V auch quadratische Untermatrizen stehen), und daß ferner den $x_1, \cdots, x_n$ sowie den $y_1, \cdots, y_n$ Systeme linearer Bedingungsgleichungen auferlegt sind. Die Arbeit betrifft nun die angenäherte Lösung solcher Gleichungssysteme, indem man die $x_1, \cdots, x_n$ willkürlich auf einen linearen Teilraum kleinerer Dimension einschränkt. Es wird gezeigt, wie man solche Teilräume so wählen kann, daß die approximative Lösung nur wenig von der richtigen Lösung abweicht; ferner wird gezeigt, wie nach Vorgabe des Teilraums die approximative Lösung durch eine Extremalforderung gewonnen werden kann. Der Vorteil der approximativen Behandlung liegt in einer Verminderung des Rechenaufwands. Verf. empfiehlt die Benutzung bei Systemen mit einer großen Anzahl von Unbekannten, um vor der umständlicheren Berechnung der exakten Lösung das Ausgangszahlenmaterial zu kontrollieren und einen vorläufigen Überblick über die zu erwartende Lösung zu gewinnen, und er macht einige Anwendungen auf Probleme des Flugzeugbaus. *A. Stöhr* (Zbl 52, 129).

Heidam, K. Zeuthen. An approximation formula for the determination of areas. Nordisk Mat. Tidskr. 3 (1955), 107–110.

The integral $J=\int_{x_0}^{x_n} F(x)dx$ is computed by means of known values $F(x_k)=y_k$ for a set of abscissae $x_0<x_1<\cdots<x_n$. To this end $F(x)$ is replaced by a third-order polynomial $f_k(x)$ for $x_{k-1}\leq x\leq x_k$. The polynomial is determined by $f_k(x_{k-1})=y_{k-1}$, $f(x_k)=y_k$; $f_k'(x_{k-1})=(y_k-y_{k-2})/(x_k-x_{k-2})$ and $f_k'(x_k)=(y_{k+1}-y_{k-1})/(x_{k+1}-x_{k-1})$ with a modification of the last two conditions for f_1 and f_n. *H. Bückner* (Schenectady, N.Y.).

Dennis, S. C. R., and Poots, G. The solution of linear differential equations. Proc. Cambridge Philos. Soc. 51 (1955), 422–432.

"In the present paper a method is suggested for obtaining solutions of linear differential equations in the form of an approximate Fourier expansion with numerical coefficients over a specified finite range of integration. By means of the appropriate Fourier transform the equation is reduced to an infinite set of linear simultaneous algebraic equations, and corresponding to each boundary condition one further equation may be derived. In some standard cases the latter are simple, and in all cases the equations may be written down exactly and solved to any desired degree of approximation by iterative methods or the relaxation method. The method is first described for second-order equations and is then extended to equations of any order. Several examples are solved, and some techniques for accelerating the convergence of iterative procedures are suggested". *E. Isaacson.*

Bahvalov, N. S. On estimation of the error in the numerical integration of differential equations by the Adams extrapolation method. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 683–686. (Russian)

The differential equation $y'=f(x, y)$ with initial conditions $y(x_0)=y_0$ is considered. Let the approximations $\bar{y}_m$ to $y_m=y(x_m)$, $x_m=x_0+mh$, be found by any given method when $m=0, 1, \cdots, k$, but when $m>k$ they are calculated by a recursion formula. The author proves a theorem that the equation $\Delta_m=\varrho_m+\beta_m$ $(m=k+1, k+2, \cdots, n)$ holds approximately for the Adams method under certain conditions, where β_m is the round-off error in the recursion formula, ϱ_m is the error inherent in the method, and $\Delta_m=\bar{y}_m-\bar{y}_{m-1}(x_m)$, $\bar{y}_m(x)$ being solutions satisfying the initial conditions. *S. Kulik* (Columbia, S.C.).

Glinskaya, N. N., and Mysovskih, I. P. On numerical solution of a boundary problem for a nonlinear ordinary differential equation. Vestnik Leningrad. Univ. 9 (1954), no. 8, 49–54. (Russian)

The authors explain how a differential equation $y''=f(x, y)$ with the boundary conditions $y(0)=y(1)=0$ can be solved by an analogue of Newton's method as elaborated by L. V. Kantorovich. Two theorems are presented without a proof: (a) on the existence of the solution of the problem, (b) estimation of the error of the approximation. *S. Kulik* (Columbia, S.C.).

Fröberg, Carl-Erik. Numerical treatment of Coulomb wave functions. Rev. Mod. Phys. 27 (1955), 399–411.

Coulomb wave functions are solutions of the differential equation $d^2y/d\varrho^2+[1-2\eta/\varrho-L(L+1)/\varrho^2]y=0$ in which η and ϱ are positive and L is a non-negative integer. The regular and irregular Coulomb wave functions are characterized by their asymptotic behavior: $F_L\sim\sin\theta_L$ and $G_L=\cos\theta_L$ as $\varrho\to\infty$, where $\theta_L=\varrho-\eta\log(2\varrho)-L\pi/2+\arg\Gamma(i\eta+L+1)$. The author describes briefly available tables of these functions, and outlines those methods and formulas which can be used to compute Coulomb wave functions to a reasonable accuracy (mostly to 5–6 digits). Under the latter heading recurrence relations, numerical integration, power series and Bessel function expansions, computation from a Riccati equation, several kinds of asymptotic expansions are mentioned. A very useful chart lists the recommended methods for various parts of the region $0<\varrho<10$, $0<\eta<10$. *A. Erdélyi.*

Garwick, Jan V. The solution of boundary-value problems by step by step methods. Arch. Math. Naturvid. 52 (1955), 95–161.

The author describes a technique for solving finite-difference equations associated with boundary-value problems for ordinary and partial differential equations. His method is nearly equivalent to an elimination method in contrast with iteration or elimination methods. He analyzes the effect of round-off error. *E. Isaacson.*

Tasny-Tschiassny, L. Nets composed of parts of circles for the approximate solution of field problems. Austral. J. Phys. 8 (1955), 8–29.

"The two-dimensional differential equation

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial\phi}{\partial y}\right)+\tau=0 \tag{1}$$

describes the current flow in a sheet of conductivity σ loaded by a transverse current density $(-\tau)$, ϕ being the electrical potential. It is known that equation (1) can be

solved approximately by a procedure in which the two-dimensional continuum is replaced by a net of straight-line bounded meshes, leading to an electrical network of conductances. The author shows that meshes bounded by "curvilinear rectangles" can be equally well dealt with and, on the basis of different conformal transformation functions for the individual meshes, derives the formulae required for a solution, if the mesh boundaries are circle arcs or circle arcs and straight lines. A good fit of the contours of the boundaries and equipotentials and their orthogonal trajectories can be obtained. This reduces the number of meshes without impairing the accuracy. Sharp corners at boundaries can be dealt with in a similar way. Formulae for a good accuracy computation of potential gradients and a method for changing the mesh size abruptly are given. Two examples using nets of only four meshes demonstrate the power of the method, the maximum errors being of the order of a few per cent." *E. Isaacson* (New York, N.Y.).

Crandall, Stephen H. An optimum implicit recurrence formula for the heat conduction equation. Quart. Appl. Math. **13** (1955), 318–320.

The author considers numerical solution of $D[\phi]\equiv\phi_t-\phi_{xx}=0$ using the finite-difference approximant

$$F[\psi]\equiv\psi_{j,k+1}-\psi_{j,k}-r[\theta\delta^2\psi_{j,k+1}+(1-\theta)\delta^2\psi_{j,k}]=0,$$

where $\psi_{j,k}\equiv\psi(j\Delta x, k\Delta t)$, $\delta^2\psi_{j,k}\equiv\psi_{j+1,k}-2\psi_{j,k}+\psi_{j-1,k}$, $\Delta x=h$, $\Delta t=rh^2$, $0\leq\theta\leq1$. What the reviewer calls the "error of approximation" is $F[\phi]=Ah^2+Bh^4+O(h^6)$, where A, B are functions of r, θ. The solution of $F[\psi]=0$ can be expressed in the form

$$\psi_{j,k}=\sum_{\alpha=1}^{M}X_j^{(\alpha)}\mu_\alpha{}^k,$$

where M is, in general, the number of mesh-points in the x-direction and μ_α (a function of r, θ) is the "growth rate" associated with the αth mode $X^{(\alpha)}$. The author plots in the (r,θ)-plane the curves $\mu_M=-1$, $\mu_M=0$ (each for $M\to\infty$, respectively the "stability-limit" and "oscillation-limit") and the curve $A=0$. He remarks that $A=B=0$ for $r=0.2236$, $\theta=0.1273$, and suggests that the corresponding difference equation is probably a very good one for integrating $D[\phi]=0$. *M. A. Hyman* (Philadelphia, Pa).

Volkov, E. A. Estimates of the error in the solution by the method of grids of Dirichlet's problem for the Laplace equation. Dokl. Akad. Nauk SSSR (N.S.) **96** (1954), 897–899. (Russian)

A number of estimates for the error of the five-point finite-difference approximation of the inner Dirichlet problem for the Laplace equation are discussed and evaluated. Several of these are based on the finite-difference approximation of the Green's function for the Laplace equation. *C. Saltzer* (Cleveland, Ohio).

Young, David M. ORDVAC solutions of the Dirichlet problem. J. Assoc. Comput. Mach. **2** (1955), 137–161.

The unit square is covered by a uniformly spaced 20×20 mesh. Under the author's direction, a code was prepared for the ORDVAC automatic computing machine which solves the Dirichlet difference problem over any subregion whose boundary is formed by mesh lines. Either the 5-point or 9-point difference approximant to Laplace's equation may be used, with $h=\Delta x=\Delta y$ an integral multiple of 1/20. The resulting simultaneous linear equations are solved by successive over-relaxation. The author presents calculations for five different regions, including one which is doubly-connected. For each region he discusses the rate of convergence as a function of the over-relaxation factor ω used. For rectangles, the optimal ω ($=\omega_b$) is known; the author suggests ways of estimating ω_b for other regions. For the unit square, he discusses the truncation error (i.e. departure of the difference solution from the differential solution) as a function of the difference equation (5-point or 9-point), of h, and of the position of discontinuities on the boundary.

M. A. Hyman (Philadelphia, Pa.).

Abramowitz, Milton, and Cahill, William F. On the vibration of a square clamped plate. J. Assoc. Comput. Mach. **2** (1955), 162–168.

The authors consider solution over the unit square of $\nabla^4u=\mu^2u$ with boundary conditions $u=\partial u/\partial n=0$, where $\partial u/\partial n$ denotes differentiation of u normal to the boundary. They use difference methods, replacing the biharmonic operator by the usual 13-point formula. Good estimates of the lowest eigenvalues μ_1, μ_2, μ_3 are available from variational calculations, and these are compared with the values now obtained by difference methods. It appears that the errors in the μ's decrease rather slowly as the mesh-width decreases; however, for a particular mesh-width, the errors in the eigenvalues can be greatly reduced (up to a point) by using higher order difference approximations to the boundary condition $\partial u/\partial n=0$.

M. A. Hyman (Philadelphia, Pa.).

Clark, Martha W. High-speed computing machine calculation of supersonic axisymmetric flows. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. no. **830** (1954), 35 pp.

Consider steady totally supersonic gas flow past a body of revolution at zero angle of attack; a shock-wave is attached to the body's pointed nose. Introducing as independent variables the characteristic coordinates α, β one seeks to solve a system of non-linear first-order partial differential equations for x, y, u, v, π, subject to certain initial conditions (Taylor-Maccoll cone flow near the nose) and boundary conditions (at the shock and at the body); the position of the shock is also to be determined. Here (x, y), (u, v), and π measure respectively the position (in an axial plane), velocity, and entropy of a gas particle with coordinates α, β. This paper describes in detail how this problem was prepared for calculation on the ORDVAC automatic computing machine. The actual computations performed were for ogive-cylinders in supersonic air streams. [It should be noted that similar calculations were carried out during 1951–53 by R. C. Roberts, B. F. Cheydleur, et al. of the U. S. Naval Ordnance Laboratory (NOL), using the machines Mark II and NORC. NOL employed a computation scheme developed by M. A. Hyman which differs markedly from either the Aberdeen scheme described here or a mechanization of one of the usual hand procedures. This NOL method, which proved both fast and accurate, has not been published (except in certain technical memos of limited circulation) but a summary may be found in Chapter 4 of Hyman's thesis [Technische Hogeschool te Delft, 1953; MR **15**, 746]; there "the method of characteristics" is discussed in detail, especially for gas-dynamical problems and from the point of view of automatic computing machines.]

M. A. Hyman (Philadelphia, Pa.).

Haque, S. M. A. Principles of numerical weather forecasting. Pakistan J. Sci. **7** (1955), 41–46.

Stesin, I. M. An estimate of the precision of computation of eigenvalues by means of continued fractions. Vyčisl. Mat. Vyčisl. Tehn. **2** (1955), 145–150. (Russian)

Let $\varkappa_1>\varkappa_2>\cdots$ be the set of numbers such that $\varkappa_i\phi_i(x)=\int_a^b K(x,y)\phi_i\,dy$, where K is a symmetric positive definite kernel. Let $f_0(x)=\sum_{i=1}^{\infty} d_i\phi_i(x)$ with $\sum d_i^2=1$. Let $f_k=\int_a^b Kf_{k-1}dy$, $c_k=\int_a^b f_k f_0 dx$. Develop the series $\sum_{k=0}^{\infty}(-1)^k c_k z^{-k-1}$ into a continued fraction

$$(*)\qquad \frac{1}{z+}\,\frac{b_1}{1+}\,\frac{b_2}{z+}\,\frac{b_3}{1+}\cdots.$$

Let the zeros of the denominator $Q_n(z)$ of the nth convergent of (*) be denoted by $z_{n1}>z_{n2}>\cdots$. It is known [Stesin, Uspehi Mat. Nauk (N.S.) **9** (1954), no. 2(60), 191–198; MR **16**, 405], that $z_{ni}\to\varkappa_i$ as $n\to\infty$. The author now gives asymptotic bounds for $h_{ni}=\varkappa_i-z_{ni}$, as $n\to\infty$. The proof for even n is given in detail. Let

$$A_{ni}=(b_n+b_{n+1})(z_{n,i-1}-z_{ni})^{-1},$$
$$B_{ni}=-Q_{n+1}(-z_{ni})\{Q_n(-z_{ni})z_{ni}\}^{-1}.$$

The result for even n is that, up to quantities of 3rd order in h_n,

$$B_{ni}(1-A_{ni})<h_{ni}<B_{ni}(1+A_{ni})(n\to\infty).$$

G. E. Forsythe (Los Angeles, Calif.).

★ **Nakada, Takashi, Ikebe, Yô, and Yamada, Tetsuo. Numerical analysis of transient phenomena.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 589–594. Science Council of Japan, Tokyo, 1952.

★ **Morita, Katsuhiko. On the high precision four variable nomographic charts of separable type.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 299–302. Science Council of Japan, Tokyo, 1953.

Morita, Katuhiko, and Simokawa, Yakiti. Nomographic solutions of ordinary differential equations of the first and second orders. Mem. Fac. Tech. Kanazawa Univ. **1** (1953), no. 2, 1–13.

If a differential equation $f(x, y, y')=0$ is equivalent to the vanishing of a Massau determinant the corresponding alignment chart can be used four times in succession, starting with the initial condition, to determine the k's for the Runge and Kutta method. The increment in y can then be determined numerically as usual. The authors discuss the application of this method to the standard first-order equations, the second-order linear equation and systems of two first-order equations. Numerical results, with comparisons, are given for special cases (including $J_1(x)$). In obtaining these solutions use was made of the first author's high precision alignment charts [J. Soc. Appl. Mech. Japan **2** (1949), 85–86, 110; MR **12**, 134] and another procedure here introduced for increasing the accuracy of results obtained from an alignment chart.

R. Church (Monterey, Calif.).

Džems-Levi, G. E. Normierte Massausche Determinanten und angenäherte Konstruktion von Nomogrammen. Moskov. Gos. Univ. Uč. Zap. **163** (1952), Mat. **6**, 133–136. (Russian)

Bisher gibt es kein praktisch befriedigendes Kriterium dafür, ob für eine allgemeine Gleichung der Form $z=F(x, y)$ ein Fluchtliniennomogramm konstruiert werden kann. Es können also nur für bestimmte Klassen von Gleichungen Fluchtlinientafeln entworfen werden. Deshalb sind Methoden für eine näherungsweise Nomographierung von Interesse. Verfasser legt eine analytische Methode zur näherungsweisen Konstruktion von Fluchtlinientafeln dar und führt für seinen Zweck den Begriff der normierten Massauschen Determinante ein. Ist es möglich für die Gleichung $z=F(x, y)$ ein Fluchtliniennomogramm zu konstruieren, so kann dieses stets projektiv so transformiert werden, daß vier gegebene Skalenpunkte allgemeiner Lage in vier gegebene Punkte der Ebene übergehen. Die letzteren vier Punkte werden Stützpunkte genannt. Es läßt sich zunächst zeigen, daß man für die Konstruktion des Nomogramms bloß zwei Funktionen zu bestimmen braucht, und nicht vier, wie in der Literatur gelegentlich angegeben wird. Es brauchen nämlich nur die beiden Funktionen bekannt zu sein, die in der Parameterdarstellung der z-Leiter vorkommen. Die Parameterdarstellungen der x- und y-Leiter lassen sich dann bereits hinschreiben. Setzt man diese Ausdrücke in die Massausche Determinante ein, so enthält diese nunmehr die beiden Koordinaten des laufenden Punktes der z-Leiter. In dieser Form nennt der Verf. die Massausche Determinante normiert. Für die angenäherte Nomographierung einer gegebenen Gleichung empfiehlt der Verf., die Lösungsleiter als von einer Anzahl von Parametern abhängig zu wählen und die Parameter dann so zu bestimmen, daß der Fehler infolge des Ersetzens der genauen Gleichung durch die in der Massauschen Determinante tatsächlich benützten eine gewisse Größe nicht übersteigt. Soll dabei im endgültigen Nomogramm die Lösungsskala geradlinig und gleichförmig sein, so hat man von einer normierten Massauschen Determinante auszugehen, deren Lösungsskala projektiv ist. In diesem Falle hat man fünf Parameter für diese Skala zur Verfügung. Dieses noch ziemlich umständliche Verfahren kann vereinfacht werden, indem man den Umstand ausnützt, daß auf den Verbindungsgeraden der Stützpunkte der x-Leiter mit denen der y-Leiter die entsprechenden z-Werte liegen müssen. Aus dieser Bedingung kann man vier der fünf Parameter bestimmen.

W. Schmidt (Zbl **49**, 211).

See also: Smiley, p. 340; Courant, p. 373; Lepskiĭ, p. 399; Linnaluoto, p. 426; Nishimura, p. 431; Livesley, p. 431; Budden, p. 434.

Tables

★ **Schütte, Karl. Index of mathematical tables from all branches of sciences. Index mathematischer Tafelwerke und Tabellen aus allen Gebieten der Naturwissenschaften.** Verlag R. Oldenbourg, München, 1955. 143 pp. DM 14.50.

This is, in some respects, a pocket version of the index of A. Fletcher, J. C. P. Miller, and L. Rosenhead [An index of mathematical tables, McGraw-Hill, New York, 1946; MR **8**, 286]. There is a need for a book giving, in a convenient form, information about reasonably accessible tables of not too specialized functions. Indeed, such a volume is at least as important as actual collections of such tables. Spot checking of the (mathematical tables part of the) volume suggests that it is a reliable guide (in so far as outstanding tables are listed), and reveals

intriguing titles which will attract the table-fan. There are however various misprints and wrong credits (e.g., the National Bureau of Standards receives many credits at the expense of individual staff members) and some entries of doubtful value or position (e.g. Farey series under Harmonic analysis). A thorough revision of this volume could make it a useful reference book.

After an introductory chapter on numerical methods, including machines and instruments, there comes the following series of chapters on mathematical tables: Logarithms; Logarithms of trigonometrical functions; Trigonometrical functions: Simple functions derived from the elementary functions; Number theory; Elementary transcendental functions; Elliptic functions and integrals, spherical and Bessel functions; Probability functions and other higher transcendental functions. There are then chapters on tables which are applicable in various fields: physics, chemistry, engineering, geodesy and geophysics, astronomy and astrophysics, meteorology and navigation (terrestrial and celestial). The concluding chapter lists various collections of tables, formulas and conversion tables. There are indices of Authors, Institute and Abbreviations.

The chapters are divided into Sections; Section titles are given in English and German. The place and date of publication of books listed are given and the number of decimal places in the tables (where this is reasonably definite) is given on the left-hand margin; no indication of interval of tabulation or of means for interpolation appear. Page numbers of tables in periodicals are not given and the title of the article is often omitted, so that only the volume number is given. Within a section the material is arranged in order of precision and within a given precision, in chronological order; space is left for addition of new entries. *John Todd* (Washington, D.C.).

Clenshaw, C. W., and Olver, F. W. J. The use of economized polynomials in mathematical tables. Proc. Cambridge Philos. Soc. **51** (1955), 614–628.

Suppose $f(x)$, tabulated at interval h, can be represented for $0\leq p\leq 1$ as

$$f_p=f(a+ph)=\alpha_0+\sum_{r=1}^{\infty}2\alpha_r T_r^*(p),$$

where $T_r^*(p)=T_r(2p-1)=\cos(n \arccos(2p-1))$. Expressions for α_n in terms of central differences of f were given by J. C. P. Miller [Proc. Roy. Soc. Edinburgh. Sect. A. **62** (1946), 204–210; MR **8**, 172].

Let the first $n+1$ terms of the series be arranged as $f_p\doteq C=c_0+c_1p+\cdots+c_np^n$, n being chosen so that the remainder is appropriately small. Then a tabulation of $c_0, c_1, \cdots, c_n$ will make interpolation easy. (Compare the case when f and its reduced derivatives $h^nf^{(n)}/n!$ are tabulated.) In general, $c_0\neq f_0$, which is inconvenient. However, the leading term of f_0-c_0 is $(-1)^n$ times the leading term in $2\alpha_{n+1}$. If $T_{n+1}^*+(-1)^n$ is approximated by a polynomial ϕ of degree n, without constant term, then we may use this to obtain, instead of C, a representation $f_p\doteq A=f_0+a_1p+\cdots+a_np^n$. The polynomial ϕ actually chosen is not the best approximation under the condition $\phi(0)=0$. An additional condition $\phi(1)=0$ is imposed, which weakens the (theoretical) approximation; it is, however, shown that this restriction improves matters when round-off realities are taken into account. Expressions for $a_1, a_2, \cdots, a_n$ are given, for $n=2(1)6$, in terms of f and its central differences.

Practical comparisons between the use of A and other methods (Bessel, Everett, Lagrange) were made. Taking account of both the size of the table and the time required for interpolation, it would appear that the present scheme is the most convenient. It is pointed out that numerical differentiation of tabulated functions and inverse interpolation can be handled with the new method.

Expansions valid for $|p|\leq\frac{1}{2}$ instead of for $0\leq p\leq 1$ were considered (cf. Miller, loc. cit.). Although advantageous for the table-maker, they have disadvantages for the table-user (e.g., blunders caused when p is negative). Their use is not advocated. Economization of Everett-type formulae is discussed briefly. *John Todd.*

Bacon, Ralph Hoyt. Integral solutions of $x^2+y^2+z^2=r^2$. School Sci. Math. **47** (1947), 155–164.

This paper lists decompositions of odd numbers into the sum of three squares for numbers $r=3(2)99$. [See the following review.] *C. B. Tompkins.*

Miksa, Francis L. A table of intergral solutions of $a^2+b^2+c^2=r^2$. Math. Teacher **48** (1955), 251–255.

Solutions of the equation of the title for $r=3(2)207$. The paper also lists 14 errata which the author has found in the table reviewed above. *C. B. Tompkins.*

Weinblum, Georg P. A systematic evaluation of Michell's integral. The David W. Taylor Model Basin, Washington, D. C. Rep. **886** (1955), vi+59 pp.

The integral of the table occurs in a theory of wave resistance of ships due to J. H. Michell [Phil. Mag. (5) **45** (1898), 106–123]. The tables, which were computed on punched card equipment at the National Bureau of Standards, facilitate the calculation of the resistance of ships of certain shapes at certain speeds. *John Todd.*

Saito, Reiko, and Morita, Masato. Clebsch-Gordan coefficients for $j_2=5/2$. Progr. Theoret. Phys. **13** (1955), 540–542.

Using a formula given on p. 75 of Condon and Shortley's "Theory of atomic spectra" [Cambridge, 1935], the author tabulates the explicit expressions of the Clebsch-Gordan coefficients $(j_1j_2m_1m_2|j_1j_2jm)$ in terms of j_1 and m for $j_2=5/2$, $j-j_1=-5/2(1)5/2$, $m_2=m-m_1=-5/2(1)5/2$.
A. Erdélyi (Pasadena, Calif.).

Worsley, Beatrice H. Solutions of a nonlinear differential equation arising in the theory of diffusion flames. Math. Tables Aids Comput. **9** (1955), 112–116.

Graphical presentation of numerical solution of

$$v''=\frac{K}{x+x_1}v(v+x+x_1-1),\quad v(0)=1-x_1,\quad \lim_{x\to\infty}v(x)=0$$

for various values of K and x_1.

See also: Benson and Schreiber, p. 300; Dingle, p. 302; Patz, p. 347; Gloden, p. 347; Sato, p. 363.

Mathematical Machines

★ **Forbes, George F. Digital differential analyzers. Part I. Elements. 2d ed.** Pacoima, Calif., 1955. v+54 pp. $ 2.50.

The booklet is intended to be an application manual for digital differential analysers. These are electronic machines with a drum memory and with counter-pairs as

integrators. Variables are represented by sequences of pulses. The integral $z=\int y dx$ is performed by adding the integrand y to an accumulator whenever a pulse for the variable x appears. The overflows of the accumulator generate the pulses of the z-sequence. After explaining the nature of such a machine in more details the author develops an introduction to the fundamental methods of operating the differential analyser. This includes scale-factors, the elementary operations of addition, subtraction, multiplication, division, and the generation of functions by means of simple set-ups. Special emphasis is on trigonometric and algebraic functions; Amble's method of inverting functions by regenerative set-ups is also dealt with. The booklet is equipped with many diagrams and with an appreciable number of exercises. *H. Bückner.*

Stiefel, Eduard. Rechenautomaten im Dienste der Technik. Erfahrungen mit dem Zuse-Rechenautomaten Z4. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen. Heft **45** (1955), 29–45; Diskussion 47–65.

This is mainly a report on the work done and the experience gathered with the low-speed, small-memory relay computer Z4 when tabulating functions, solving polynomial equations, linear and non-linear differential equations, partial differential equations, eigenvalue problems, etc. The author emphasizes that a general-purpose computer, used for problems of various length and complexity, will be useful only if its speed is matched by adequate in- and output facilities and if programming is as easy as possible. This has lead to the following requirements for the new machine (ERMETH) replacing Z4: (i) ERMETH is a decimal machine throughout with floating point arithmetics. (ii) The "coding box" on which codes will be punched will show the conventional mathematical symbols (+, −, ×, ÷ etc.). (iii) A few simple orders together with a "Small Manual" will suffice for the simple computations of "novices", but additional facilities will be available for "experts". After the lecture, H. Behnke, L. Brandt, G. Hoheisel, H. Kaiser, J. von Neumann, E. Sperner, E. Peschl, H. Peterson, R. Sauer, E. Stiefel, A. Walther and W. Weizel discussed various aspects ranging from storage devices and analogy machines to economics and training of mathematical personnel. *H. H. Goldstine* (Princeton, N.J.).

★ **Komamiya, Yasuo, and Suekane, Ryota. On the pilot model of Electrotechnical Laboratory automatic relay computer.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 333–337. Science Council of Japan, Tokyo, 1954.

Sternberg, Sidney. An accurate electronic multiplier. RCA Rev. **16** (1955), 618–634.

Householder, Alston S. The generation of error in digital computation. Oak Ridge National Laboratory, Oak Ridge, Tenn., Rep. **ORNL 1983** (1955), 79 pp.

In a high-speed digital machine such as ORACLE, the Oak Ridge National Laboratory computer, a binary number of the form

$$b=-\beta_0+\beta_1\cdot 2^{-1}+\beta_2\cdot 2^{-2}+\cdots \quad (\beta_r=0 \text{ or } 1),$$

with a possible infinity of terms, is represented digitally by the truncated form

$$b^*=-\beta_0+\beta_1\cdot 2^{-1}+\beta_2\cdot 2^{-2}+\cdots+\beta_s\cdot 2^{-s},$$

or the rounded form

$$b_r^*=(b+\varepsilon),\ \varepsilon=2^{-s-1}.$$

The use of such digital numbers gives rise to errors in calculation, and the purpose of the paper is to provide estimates for the upper and lower error bounds in various computing operations.

Such estimates are first discussed and obtained for the elementary basic operations of multiplication and division, the evaluation of ab/c, and for three ways of calculating a/bc. The results are applied to the determination of powers of numbers, and to two ways of evaluating polynomials. A special case is the computation of a polynomial as the truncated form of an infinite series, and detailed attention is given to the series for e^x, $\sin x$, $\cos x$ and $\log x$. The use of continued fractions is suggested, and uncertainties given, for $\tanh x$, $\tan^{-1} x$, $\log(1+x)$ and e^{-x}. The reverse process of finding a root, by the binary equivalent of Horner's process, and by Newton's method applied particularly to $\sqrt{a}$, are also analysed.

The paper ends with a more complicated analysis of the error involved in the solution of algebraic equations and in some other matrix operations, and of the uncertainty in an eigen-value found (i) by the matrix iteration method and (ii) by the "rotations" method of Givens. *L. Fox* (Teddington).

Bodewig, E. Die Berechnung der ganzzahligen Adjungierten. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indagationes Math. **17** (1955), 91–94.

It is proposed to calculate the exact inverse of a given matrix with integral elements by applying the Smith algorithm for diagonalization. The method is clearly explained and an illustrative example is worked, yet the essential difficulties of the method, which involves finding greatest common divisors, are inadequately considered. In particular, the reviewer could not verify the count of necessary multiplications, and this figure seems in any event to be meaningless for modern digital computers in cases such as this where the number of digits in the numbers may grow large. *W. Givens.*

Reitwiesner, George W. Two machine methods for determination of eigenvalues and eigenvectors of real symmetric matrices. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. no. **850** (1954), 23 pp.

The two methods of computation referred to in the title are the Jacobi reduction to diagonal form by iteration of a sequence of orthogonal congruences, each involving only two rows and columns, and a related method due to the reviewer and leading initially to a "tridiagonal" (Jacobi) form [Oak Ridge National Laboratory Rep. ORNL **1574** (1954); MR **16**, 177]. The first of these methods was coded for orders $\leqq 39$ for the Ordvac (using a partitioning procedure for the vectors) and for orders $\leqq 15$ or $\leqq 29$ (if only eigenvalues are wanted) for the Edvac. Applications are reported to: 1) factor analysis; 2) the asymmetric rotor, where the eigenvalues of 1,313 matrices of orders $\leqq 7$ where computed; and 3) a probability problem leading to the matrix with elements the signed binomial coefficients $a_{ij}=(-1)^{i-j}C_{r-(i-j)}^{2r}$, where the eigenvalues for $r=1, 2, \cdots, 10$ and orders $\leqq 29$ have been computed. The number of iterations required in each case of 3) is tabulated fully. Since all the matrices of 2) were already in the Jacobi form and those of 3) were strongly diagonal, the problems would be especially suited to the other method described. *W. Givens* (Knoxville, Tenn.).

Shannon, C. E. Game playing machines. J. Franklin Inst. **260** (1955), 447–453.

Curti, P., et Dubois, F. Mécanisation des calculs de la balistique extérieure. Mém. Artill. Franç. **29** (1955), 971–983.

★**Draper, Charles Stark, McKay, Walter, and Lees, Sidney. Instrument engineering. Vol. III. Applications of the instrument engineering method. Part 1. Measurement systems.** McGraw-Hill Book Company, Inc., New York, 1955. xxviii+879 pp. $17.50.

The book covers various aspects of measuring instruments, especially for temperature, pressure and vibrations. It is intended to be a textbook and reference for students and engineers. A special chapter is devoted to integrators and differentiators and this review will be exclusively concerned with these two mathematical instruments. Various technical forms are covered. These include ball and disc integrators, wattmeters, liquid-flow integrators, thermal integrators, pneumatic pressure integrators and especially capacitance integrators. The differentiators cover mechanical and electrical models as well as various mechanical and electrical tachometers. It seems that the goal of the chapter is to normalize definitions and denotations and to give a description of the performance of the instruments, especially by means of self-defining symbols. A considerable number of excellent illustrative pictorial diagrams together with functional diagrams are presented. Some general remarks on how to use the instruments in differential analyzer setups are also included. While this reviewer appreciates the careful technological treatment of the subject, he still wonders whether self-defining symbols with subscripts of twenty characters, or even more, will appeal to everybody concerned with questions of standardization and references of integrators and differentiators. *H. Bückner.*

Gonzalez del Valle, A. Le calculateur analogique C. A. C. Calc. Automat. y Cibernet. **4** (1955), no. 11, 18–23. (Spanish summary)

Kobrinski, N. E. Machines for integration of systems of ordinary differential equations. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **9** (1955), no. 2, 5–57 (6 plates). (Romanian)

Translation of ch. 6 of Kobrinskiĭ's Mathematical machines with continuous action [Gostehizdat, Moscow, 1954; MR **16**, 866].

Nicolau, Edmond. Un nouveau système différentiel non linéaire étudié par voie électronique. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 465–475. (Romanian. Russian and French summaries)

Ablow, C. M., and Brigham, Georges. An analog solution of programming problems. J. Operations Res. Soc. Amer. **3** (1955), 388–394.

Kurakin, K. I. On the choice of optimal characteristics of linear differentiators in systems of automatic regulation. Avtomat. i Telemeh. **16** (1955), 293–299. (Russian)

Using the least-squares criterion of Wiener's theory of filtering and prediction, the optimal characteristic of a linear differentiator in the presence of noise is determined.

N. Levinson (Cambridge, Mass.).

Eremeev, N. V. Apparatus for sketching a graph of the velocity of the slider of a crank-connecting-rod mechanism. Moskov. Gos. Univ. Uč. Zap. **172** (1954). Meh. **5**, 245–249. (Russian)

Artobolevskiĭ, I. I. Mechanisms for drawing conics with envelopment. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 702–705. (Russian)

See also: Bodewig, p. 339; Foster and Teichroew, p. 382; Greenstadt, p. 411; Young, p. 413; Edamoto, p. 429; Mealy, p. 436.

ASTRONOMY

★**Hil'mi, G. F. Problema *n* tel v nebesnoĭ mehanike i kosmogonii. [The problem of *n* bodies in celestial mechanics and cosmogony.]** Izdat. Akad. Nauk SSSR, Moscow, 1951. 155 pp. 6.50 rubles.

This volume may be considered as a continuation of Birkhoff's "Dynamical systems" [Amer. Math. Soc. Colloq. Publ., vol. 9, New York, 1927], in that it studies the n-body problem from the point of view of paths in a high-dimensional phase space. Some striking new results are presented and various conclusions are drawn as to the long-term behavior of stellar and interstellar matter, in particular with reference to formation of the solar system. There is much emphasis on the role of collisions as "energy sinks" which help to stabilize the motion. The following is a summary of the contents.

Chapter 1: Equations, general integrals and basic inequalities of the n-body problem. Sundman's inequality [see Birkhoff, loc. cit. p. 267] and others due to the author are presented, in particular one governing the rate of growth of the minimum distance ϱ between the n bodies.

Chapter 2: Some problems of general dynamics. The flow in an n-dimensional phase space is described; $f(A, t)$ denotes the set obtained from A following the flow through time t. Invariant and recurrent sets are defined and studied and their measurability established. A set A is recurrent for r tending to $+\infty$ if $f(A, t)$ meets A for arbitrarily large t; a point p is recurrent if $f(p, t)$ meets each ε-neighborhood of p for arbitrarily large t. It is shown that if a set M has positive measure and contains no recurrent points, then M contains a bounded subset M_1 of positive measure which is nonrecurrent. The results are applied to the n-body problem. The set Ξ^+ [set Ξ] is defined as the set of all p for which $f(p, t)$ is defined for $0 \leqq t < \infty$ [all t]. It is shown that almost all points of Ξ^+ give rise to paths which are either unstable for t tending to $+\infty$ or are such that on the path the potential energy is "oscillatory."

Chapter 3: Formation and dissolution of stable subsystems of systems of gravitational particles. It is shown that when the total energy H is negative, ϱ is bounded above, but if $H>0$, then the maximum distance R between particles must approach ∞ as $t \to \infty$. If the initial conditions satisfy certain inequalities, one can conclude that ϱ itself tends to ∞ as t tends to ∞. For the 3-body problem inequalities for the initial conditions are given

which guarantee that one body recedes to ∞ as $t\to\infty$, while the distance between the other two remains bounded. With the aid of these theorems and a numerical integration it is shown by an example of Schmidt that the phenomenon of "capture" can take place; i.e., one body can approach a second one from infinite distance (in the past) and then remain at bounded distance from the second as t tends to ∞. From the existence of this example it is inferred that capture has a non-zero probability in general.

Chapter 4: Systems of gravitational bodies with initial conditions of nonmechanical origin. It is remarked that if the motion of n bodies, considered as physical bodies, is followed backwards in time, it may happen that collisions arise. In such a case the present condition is considered to be of nonmechanical origin. It is shown that almost all points of the set Q, consisting of all points p in Ξ for which R tends to ∞ as t tends to ∞ and R does not tend to ∞ for t tending to $-\infty$, give rise to motions for which ϱ has lower bound 0 for $t\leqq 0$, so that collisions would be inevitable in the past.

Chapter 5: Evolution of systems of gravitational bodies under the influence of nonelastic collisions. The notion of recurrence is generalized to "recurrence in probability" and it is shown that all bounded motions in the phase space have this property. The notion of a minimal core of a motion is introduced; roughly this is a closed set W, minimal with respect to the property that the motion spends most of its time in the neighborhood of W. It is shown that every bounded motion has such a minimal core. The theory is applied to a cloud of meteoric dust, for which coalescing of particles is permitted. It is shown that if such a cloud is stable and has non-zero angular momentum, and if a sufficient amount of mechanical energy is lost (to heat) through collisions, then in the cloud there will be a tendency to condensation, either in a decrease in the average distance between the particles or in the formation of large particles through coalescing of small ones; if enough mechanical energy is lost, the large particles will acquire considerable angular momentum.

Chapter 6: General questions. The importance of loss of mechanical energy for the condensation process is emphasized. Reference is made to philosophical speculations of Engels and to the views of Newton on formation of the solar system. The Schmidt theory on evolution of the solar system is described. According to this theory the sun acquired many particles of dust by capture and these then condensed, according to the model of Chapter 5, to form planets; the theory accounts qualitatively for the present angular moments of the planets.

W. Kaplan (Ann Arbor, Mich.).

Agostinelli, Cataldo. Su una soluzione periodica del problema ristretto dei tre corpi più generale di quella di Hill. Univ. e Politec. Torino. Rend. Sem. Mat. **13** (1954), 131–151.

We have here an existence proof and numerical calculation for the periodic solution of the lunar problem after the manner of Hill's famous work but without complete neglect of the solar parallax. Namely, the present author retains the terms of the first order in the parallax. He presents the solution in the form of series of cosines and sines of multiples of $(n_0-n)t$, n being the mean motion of the planet about the sun and n_0 being the arbitrary constant corresponding to the mean motion of the satellite. He gives formulas permitting the successive calculation of the coefficients of these series with arbitrarily good accuracy and computes explicitly the first six of them. A qualitative result of taking into account the parallax is to lose part of the symmetry of Hill's solution, which is symmetrical with respect to both of two suitably chosen rotating axes. In the present solution we have symmetry only with respect to one axis. *D. C. Lewis.*

Steffensen, J. F. On the differential equations of Hill in the theory of the motion of the moon. Acta Math. **93** (1955), 169–177.

In an earlier paper [Oversigt Danske Vid. Selsk. Forh. **1909**, 319–335, pp. 320–323] the author transformed Hill's equations, introducing polar coordinates (r, l) instead of rectangular coordinates (p, q), putting $\varepsilon=1/r^3$, $\varrho=r^{-1}dr/dt$, $\omega=1+dl/dt$, and eliminating the time t. To remove the trigonometric functions from these transformed equations, he puts

$$\cos 2l=x,\ \omega=1+\eta,\ \varrho=\zeta\sin 2l,$$

and obtains a system of three first-order differential equations with respect to ε, ϱ, η, the independent variable being x. Further, it is shown that this last system can be satisfied by power series in x which converge for sufficiently small $|x|$. The coefficients of these series are determined by recurrence formulas. The chief advantage of the present method compared with that in the paper cited above is that the coefficients of the power series can be calculated much easier than the corresponding coefficients of the trigonometric series in the earlier paper. In the given numerical example the convergence of the series is equally good in both methods. *E. Leimanis.*

Burgers, J. M. Rotational motion of a sphere subject to visco-elastic deformation. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. B. **58** (1955), 219–226, 227–235, 236–237.

A body of spherical shape is made to rotate about a fixed axis with constant angular velocity. The centrifugal deformation is opposed by the gravitational self-attraction of the body and also by internal stiffness and viscosity. A mechanical model for the coupling of stiffness and viscosity is used to obtain an expression for the difference between the moments of inertia of the spheroidal body about its polar axis and one of its equatorial axes. The Euler equations of motion of a body whose moments and products of inertia are subjected to small changes are obtained. Using various types of approximations, the following results are obtained: (a) for small oscillations of the axis of rotation about a fixed direction, a very large stiffness leads to the nutational frequency of a completely rigid sphere, but if the stiffness is lessened, the frequency decreases too; (b) the body has the same type of stability or instability as has an ordinary rigid sphere. The effect of small changes in the moments of inertia of the deformable body, other than those due to the centrifugal force, are next considered. It is shown that large polar wandering can occur and the speed of wandering is in conformity with Gold's result. Both the moments and the products of inertia are then varied by small, but arbitrary, amounts; these amounts are constant in one case and slowly varying functions of the time in another. The methods of dealing with these more complicated conditions are indicated.

G. C. McVittie (Urbana, Ill.).

Garfinkel, Boris. The mean daily insolation. Astr. J. **58** (1953), 196–199.

In this paper the author investigates the mean daily

energy F received from the sun on the surface of a planet with zero orbital eccentricity. This so called insolation is a function of the latitude ϕ on the planet and the declination δ of the sun. It is shown that the curves $F(\phi, \delta=\text{constant})$ have an interior minimum and maximum if $|\delta|<25°$. The curves $F(\phi=\text{constant}, \delta)$ have a maximum if $|\phi|<y(i)$, a maximum and a minimum if $y(i)<|\phi|<25°$ and if $i>53°3$. In these expressions $y(i)$ is a function of the obliquity i. *D. Brouwer* (New Haven, Conn.).

Garfinkel, Boris. The mean annual insolation. Astr. J. 58 (1953), 200–205.

The mean annual insolation f of a spherical planet is a function of the colatitude ψ and the obliquity i. The geometry of the surface $f(\psi, i)$ is investigated in detail. The behavior of the curves $f(\psi, i=\text{constant})$ is studied. For $0°\leqq i\leqq 45°$, $f(\psi)$ is an increasing function of ψ and for $65°5\leqq i\leqq 90°$ it is a decreasing function of ψ. For the region $45°<i<65°5$ the function has a single minimum. This last statement was already emphasized by Hargreaves [Trans. Cambridge Philos. Soc. **16** (1896), 58–94], but is rigorously proven by the author in this paper.

In the first paragraph of this paper the author states that the mean annual insolation is the principal determinant of climate. It should be borne in mind that changes in insolation from geometrical changes only cannot account for the large changes in climate like the ice ages in the pleistocene. *D. Brouwer* (New Haven, Conn.).

Lindblad, Bertil. Star-streaming and spiral structure. Stockholms Obs. Ann. **18** (1955), no. 6, 20 pp.

Nearly circular orbits of mass points in an axisymmetric field with a plane of symmetry are considered. It is shown that in the frame of reference rotating with the angular velocity, $\omega-[\omega(\omega-A)]^{\frac{1}{2}}$, where ω is the angular velocity of a circular orbit (with the same constant of angular momentum as the orbit considered) and $A=-\frac{1}{2}R\partial\omega/\partial R$ (where R is the distance from the axis) the orbits are closed symmetrical curves. The effect of perturbations on these orbits is considered; and in terms of such effects the author once again attempts to interpret the spiral structure of galaxies in the manner he has made familiar in his earlier papers. *S. Chandrasekhar* (Williams Bay, Wis.).

Ewart, D. G. On the relation between the stream and the ellipsoid constants. Monthly Not. Roy. Astr. Soc. **115** (1955), 47–56.

Distributions of peculiar stellar velocities (for groups of stars selected according to a particular criterion, such as spectral type) are usually represented by one of two models: the two-stream model of Kapteyn, and the ellipsoidal model of K. Schwarzschild. Both models represent observed distributions fairly well [for a complete discussion, see R. J. Trumpler and H. F. Weaver, Statistical astronomy, Univ. of California Press, 1953, pp. 272–280] but the ellipsoidal model is the more convenient in practice, and also has the advantage of being closely related to the dynamical theory of galactic structure.

On Schwarzschild's model, the normalized distribution function is given by

$$f_1(u, v, w)=\pi^{-3/2}kh_1^2\exp\left[-k^2u^2-h_1^2(v^2+w^2)\right],$$

and on Kapteyn's model by

$$f_2(u, v, w)=\tfrac{1}{2}\pi^{-3/2}h_2^3\{\exp\left[-h_2^2((u-q)^2+v^2+w^2)\right]+\exp\left[-h_2^2((u+q)^2+v^2+w^2)\right]\},$$

if the two streams have the same spatial density and the same dispersion in peculiar velocity. In these formulae, u, v, w, denote the cartesian components of the peculiar velocity, the u-axis having been chosen to coincide with the axis of star-streaming; h_1 and k are the two distinct axes of the velocity ellipsoid; and q is half the relative speed of the two streams.

The two-dimensional distribution functions for v and w (i.e., the original distribution functions integrated over u) will evidently be identical if and only if $h_1=h_2=h$ (say). The author shows that if one also requires the two velocity distributions to have equal mean speeds in the positive u-direction, then q, h and k are connected by the relation

$$h/k=\exp(-h^2q^2)+2hq\int_0^{hq}\exp(-x^2)dx.$$

He also shows that this relation is satisfied by values of h, k, and q, derived from various analyses of observed stellar motions.

[It should perhaps be mentioned that the above requirement is by no means unique. Thus the requirement that the mean square speeds in the u-direction be equal leads to the simple condition

$$(h/k)^2=1+2h^2q^2,$$

which is the asymptotic form of the author's requirement for small hq.] *D. Layzer* (Cambridge, Mass.).

Brahmachary, R. L. Axially symmetric solution in problems of galactic magnetic fields and a new type of red shift. Nuovo Cimento (10) **2** (1955), 850–853.

Brahmachary, R. L. On the cosmological implication of galactic magnetic fields. II. Nuovo Cimento (10) **2** (1955), 149–151.

Neyman, Jerzy, and Scott, Elizabeth L. On the inapplicability of the theory of fluctuations to galaxies. Astr. J. **60** (1955), 33–38.

The authors draw attention to a basic defect in a theory proposed by Limber [Astrophys. J. **117** (1953), 134–144; **119** (1954), 655–681; MR **14**, 803; **15**, 994] for the analysis of galaxy counts. Let the random variable $N(\omega)$ represent the number of images of galaxies in a region ω of area ω on a particular photographic plate. On Limber's theory, which derives from a more general theory of fluctuations developed by Chandrasekhar and Münch [ibid. **115** (1952), 103–123; MR **13**, 786],

$$\overline{N(\omega)}=A\omega,\ \overline{N^2(\omega)}=B\omega^2,\ \overline{N(\omega)N(\omega')}=C(\varphi)\omega^2,$$

where A, B, and C, are positive quantities, and φ is the angular distance between the centers of region ω and ω', the areas of the two regions being equal. Hence

$$\overline{N(\omega)}>\overline{N^2(\omega)}\ \text{ if }\ \omega<A/B,$$

which is a contradiction, since the mean square number of images must always exceed the mean number. Moreover, the correlation

$$\overline{N(\omega)N(\omega')}/\overline{[N(\omega)-N(\omega)]^2}=C(\varphi)/(B-A^2),$$

is independent of the area ω on Limber's theory, which is also a contradiction, since for a fixed separation between two regions of equal areas the correlation must approach zero with the area. Both contradictions stem from the absence of a term linear in ω in Limber's expression for the mean square number of galaxies.

A detailed discussion of Limber's analysis shows that his results provide a rather indirect measure of the dimensions of his réseau, rather than of any aspect of the distribution of galaxies in space. *D. Layzer.*

Neyman, Jerzy. Sur la théorie probabiliste des amas de galaxies et la vérification de l'hypothèse de l'expansion de l'univers. Ann. Inst. H. Poincaré **14** (1955), 201–244.

In a previous paper [Astrophys. J. **116** (1952), 144–163; MR **14**, 803], the author and E. L. Scott constructed a theoretical model for the spatial distribution of galaxies, taking into account the expansion of the universe only in so far as it affects the relation between the apparent and the absolute magnitude of a galaxy. In Part I of the present paper, the author has sought to reformulate the theory so as to take fuller account of the expansion. From a mathematical point of view, the modifications are not of a fundamental character; but the reader who is familiar with relativistic cosmology will probably find them inadequate.

Part II deals with the question: To what extent do clusters of galaxies interpenetrate on the Neyman-Scott model? The author derives generating functions for three random variables that characterize, in different ways, the degree of interpretation. (i) The number of galaxies in the cluster nearest a given cluster that penetrate the given cluster 'to a depth k.' (A galaxy is said to penetrate a given cluster to a depth k if it lies within a sphere whose center coincides with the center of the given cluster and whose radius is equal to the radial distance of the kth most distant galaxy in the given cluster.) (ii) The number of clusters that contain at least one member that penetrates a given cluster to a depth k. (iii) The number of clusters all of whose members penetrate the given cluster to a depth k. With the help of the generating functions for these random variables, the author derives, in the usual way, formulae for various probabilities and expectation values, such as the probability that a cluster with N or more members will not wholly contain any other cluster, the expectation value for the number of galaxies in the cluster nearest a given cluster with N or more members that penetrate the given cluster to a depth k, and so on. He also gives the results of numerical calculations based on these formulae, and performed under the supervision of E. L. Scott. These show in detail how the degree of interpenetration depends on the compactness of the clusters. *D. Layzer.*

Raychaudhuri, Amalkumar. Perturbed cosmological models. Z. Astrophys. **37** (1955), 103–107.

The author extends the work of Bonnor [same Z. **35** (1954), 10–20; MR **17**, 202] on perturbations of the Friedmann universe in which (i) there is spherical symmetry, (ii) the pressure is isotropic, (iii) matter moves on streamlines, (iv) the perturbation begins with a discontinuity across a space-like 3-surface on which the Synge-O'Brien conditions are satisfied. He finds, in agreement with Lifshitz [Acad. Sci. USSR. J. Phys. **10** (1946), 116–129; MR **8**, 175], that the perturbations develop too slowly to permit the formation of nebular condensations in the available time, and concludes that his assumptions are too restrictive. *F. A. E. Pirani.*

RELATIVITY

★**Ρωμανας, Γεωργιος. Εισαγωγη εις την οπτικην επι κινουμενων συστηματων. [Romanas, Georgios. Introduction to the optics of moving systems.]** Athens, 1954. 237 pp. (6 plates).

This is a monograph (privately published) on the propagation of light in moving bodies, based on the existence of the aether. The purpose of the author is to explain phenomena of the aberration of light from the stars and the classical experiments of Fizeau, Trouton and Noble and of Michelson-Morley on the idea of the existence of the aether. The results and interpretations of these experiments are based on the galilean theorem of addition of relative velocities. The author takes great pain to show the reader (but not convincingly in the opinion of the reviewer) the correct explanation of the above mentioned experimental facts; and at the same time disputes the validity of the theory of relativity of Einstein by claiming that it is inconsistent with the general principles of physics (classical), mechanics and classical logic. [It is the opinion of the reviewer, that the book contains very little material of scientific value.] *N. Chako.*

Pinl, M. Begriff und Ziel der einheitlichen Feldtheorien. Math.-Phys. Semesterber. **4** (1955), 183–194.

Expository paper.

Ghosh, N. N. On the solution of Γ's for a type of non-symmetric tensor field $g_{\mu\nu}$. Progr. Theoret. Phys. **13** (1955), 587–593.

The connection $\Gamma_{\lambda\mu}{}^{\nu}$ of Einstein's unified field theory is

$$\Gamma_{\lambda\mu}{}^{\nu}=\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}+S_{\lambda\mu}{}^{\nu}+2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}h_{\mu)\beta} \tag{1}$$

where $S_{\lambda\mu}{}^{\nu}=\Gamma_{[\lambda\mu]}{}^{\nu}$ is the solution of

$$2S_{\alpha\beta\gamma}X_{\alpha\mu\lambda}{}^{\alpha\beta\gamma}=K_{\alpha\mu\lambda} \tag{2}$$

[for these equations see Hlavatý, J. Rational Mech. Anal. **2** (1953), 1–52; MR **14**, 505]. The author solves a system equivalent to (2) for a very special case of the first class and computes also the symmetric part of $\Gamma_{\lambda\mu}{}^{\nu}$. This solution simplifies considerably if one restricts still more the generality of the problem. The author imposes a condition on $g_{\lambda\mu}$ which amounts to $\Delta_{(\omega}k_{\lambda)\mu}=0$.

Notes of the reviewer: I. The solution of (2) for general n has been given in the above paper of the reviewer. For $n=4$ the solution for all three classes and even for degenerate cases was exhibited by Hlavatý [ibid. **4** (1955), 247–277, 653–679; MR **16**, 870; **17**, 200]. II. It usually escapes attention that unlike in general relativity the requirement of a special form of $g_{\lambda\mu}$ in the unified theory imposes heavy conditions on $g_{\lambda\mu}$. This is due to the fact that in the unified theory we have a set of preferred null vectors, i.e. the eigen vectors of $k_{\lambda}{}^{\nu}$. *V. Hlavatý.*

Anderson, James L., and Merat, Parviz. Convervation of charge in Einstein's generalization of gravitation theory. Phys. Rev. (2) **99** (1955), 1871–1873.

It is pointed out that Einstein's λ-invariance [The meaning of relativity, 4th ed., Princeton, 1953, Appendix II; MR **14**, 805] may be built into his 1953 unified field theory from the outset by introducing a 4-potential ϕ_{λ} such that $g^{\underset{\sim}{\nu\sigma}}=\eta^{\underset{\sim}{\nu\sigma\lambda\mu}}\phi_{\lambda,\mu}$ ($\eta^{\underset{\sim}{\nu\sigma\lambda\mu}}$ is the alternating tensor density). The vector density $\eta^{\underset{\sim}{\nu\sigma\lambda\mu}}R_{\underset{\sim}{\sigma\mu},\mu}$ is identified with 4-current; it is conserved provided that the theory is invariant under gauge-transformation of ϕ_{λ}. The authors

discuss the physical difficulties of these proposals; the chief is that the 4-current vanishes wherever the field equations are satisfied. *F. A. E. Pirani* (London).

Reulos, René. Compléments à la théorie de la relativité. C. R. Acad. Sci. Paris **241** (1955), 1107.

McCrea, W. H. On the significance of newtonian cosmology. Astr. J. **60** (1955), 271–274.

It is proved that the strictly Newtonian cosmological models obtained by Milne and McCrea are valid for spherically symmetric distributions of matter that are bounded, though of arbitrarily large extent. The extension of the theory to unbounded distributions necessarily requires some modification of the meaning of Newtonian gravitation. This may be found in the assertion that the field of force is not uniquely determined by the instantaneous density distribution. *G. C. McVittie.*

MECHANICS

Marinescu, G. Sur une mécanique générale. IV. Une relation d'équivalence. Com. Acad. R. P. Române **1** (1951), 141–145. (Romanian. Russian and French summaries)

Marinescu, G. Structures algébriques sur lattis et mécanique latticielle. Com. Acad. R. P. Române **2** (1952), 109–112. (Romanian. Russian and French summaries)

The author develops further his lattice-theoretic generalization of kinematics, in which "bodies" are treated as elements of a lattice, instead of as continua or mass-points [see Acad. R. P. Române Bul. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 211–217, 575–581, 583–587; MR **13**, 662]. *G. Birkhoff* (Cambridge, Mass.).

Froda, Al. L'accélération en mécanique rationnelle. Com. Acad. R. P. Române **1** (1951), 51–55. (Romanian. Russian and French summaries)

This note rests on the author's previous work and uses its terms [especially Acad. R. P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3** (1951), 157–175; Acad. R. P. Române. Stud. Cerc. Mat. **3** (1952), 321–365; MR **15**, 692; **16**, 635]. The author gives reasons indicating that the "acceleration" which enters the law of inertia should be the "direct acceleration", rather than any of various possible substitutes. *C. Truesdell* (Bloomington, Ind.).

Callahan, Francis P., Jr. Approximately planar rigid body motion. J. Rational Mech. Anal. **4** (1955), 569–578.

The paper advocates the use of the approximate equations,

$$(1)\quad iA\ddot{Z}+Cr\dot{Z}=K,\quad K=(L+iM)\exp(i\alpha)\quad (i^2=-1),$$

where A is the equatorial, C the axial moment of inertia at the fixed point O on the axis of a symmetric gyroscope, s its spin, L and M the projections on the body-bound axes of the applied torque about O, and (according to a personal communication),

$$x+iy=Z=\sin\theta\exp(i\phi)\quad(\alpha=\phi+\psi),$$

where θ, ϕ, ψ are the Euler angles of nutation, precession, and axial revolution (not used in the paper). The argument starts with an approximate representation of a finite rotation in terms of x, y, and α, and proceeds with several assumptions of negligibility, including differentiation of approximate equations. The relative error of (1) is not stated in the paper. For $K\neq 0$, the reviewer believes that it does not exceed the modulus of

$$[iA\dot{e}+Cse-Ar(\dot{Z}+e)]/K,\quad e=\dot{\theta}(1-\cos\theta)\exp(i\phi).$$

In the case $K=0$, after a proper choice of the fixed reference frame, (1) yields $Z=\sin\theta\exp(iCst/A)$ while the exact solution (free precession) is $Z=\sin\theta\exp(iCst/A\cos\theta)$; the relative error here is of the order of θ^2 (θ being the constant angle of nutation). If a relative error of the order of θ^2 is acceptable, K approximated the projection of the applied torque at O on the fixed equatorial plane, and the angle ϕ seems redundant to this reviewer. Moreover, (1) will be advantageous if K happens to be analytic in Z and $\dot{Z}$. Otherwise, less elaborate linearizations [e.g., E. J. Routh, Advanced rigid dynamics, 6th ed., Macmillan, London, 1930, art. 214] are available even for non-symmetric rigid bodies. *A. W. Wundheiler.*

Trent, Horace M. Isomorphisms between oriented linear graphs and lumped physical systems. J. Acoust. Soc. Amer. **27** (1955), 500–527.

A generalized formulation of the relations between lumped-parameter systems and oriented graphs is presented and applied to the analysis of physical systems and the design of analogous systems. *C. Saltzer.*

Gran Olsson, R. On the analogy of some problems of dynamics of mass points and rigid bodies. II. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 46–53.

Using the results of Part I [same Forh. **27** (1954), 14–21; MR **16**, 295] the author solves the Carathéodory example of the non-holonomic motion of a sleigh, allowing, in addition, a constant transverse velocity at the point of application of the transverse force. The second "=" in equation (5.2) should be "+"; there is also a minor misprint following equation (5.44). *H. D. Block.*

Haimovici, Adolf. Contribution à la mécanique du point de masse variable. Rev. Math. Phys. **2** (1954), 26–32 (1955).

A translation into French of Acad. R.P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 61–68 [MR **15**, 836]. *R. A. Rankin* (Glasgow).

Storchi, Edoardo. Su una nuova interpretazione del principio dell'azione potenziale. Boll. Un. Mat. Ital. (3) **10** (1955), 161–165.

The author's principle of stationary potential action [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14** (1953), 771–778; MR **15**, 753] is interpreted geometrically. The free particle is taken as an example. *F. A. E. Pirani* (London).

Myasnikov, P. V. A new method of finding a class of integrable cases of motion from the general problem of rotation of a heavy rigid body about a fixed point. Moskov. Gos. Univ. Uč. Zap. **172** (1954), Meh. **5**, 143–162. (Russian)

The paper surveys all the cases of integrability in which the angular momentum has a constant projection on the

line joining the fixed point with the mass center, the latter lying in one of the principal planes of inertia. All the cases found are known. *A. W. Wundheiler.*

Bradistilov, G. The position of a triple mathematical pendulum in a plane in its periodic and asymptotic motions about an equilibrium position. Prikl. Mat. Meh. **19** (1955), 485–492. (Russian)

The author extends his previous investigations [Prikl. Mat. Meh. **19** (1955), 113–118; MR **16**, 874] to the cases when some of the pendulums are inverted in the position of equilibrium. *A. W. Wundheiler* (Chicago, Ill.).

See also: Bieniek, p. 366; Blehman and Džanelidze, p. 369; Hil'mi, p. 417.

Fluid Mechanics, Acoustics

Ertel, Hans. Ein neues Wirbel-Theorem der Hydrodynamik. S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1954**, no. 5, 12 pp. (1955).

For any continuous medium, one of the numerous equivalent forms of the general equation of diffusion of vorticity is

$$(*) \qquad \frac{D}{Dt}\int \psi \mathbf{w}\cdot d\mathbf{s} - \int \frac{D\psi}{Dt}\mathbf{w}\cdot d\mathbf{s} = \int \psi \operatorname{curl}\mathbf{a}\cdot d\mathbf{s},$$

where $\mathbf{w}$ is the vorticity, $\mathbf{a}$ is the acceleration, ψ is an arbitrary quantity, and the integrals are taken over a material surface. This is easy to establish directly; alternatively, it may be gotten by integrating the reviewer's generalization [Z. Angew. Math. Phys. **2** (1951), 109–114; MR **12**, 761] of a vorticity theorem previously discovered by the author. The author obtains the special case of (*) appropriate to perfect fluids subject to no extrinsic force and notes some further specializations. *C. A. Truesdell* (Bloomington, Ind.).

Sagomonyan, A. Ya., and Bagdoev, A. G. Penetration of a cone into a fluid with a free surface. Vestnik Moskov. Univ. **10** (1955), no. 8, 47–52. (Russian)

The problem of vertical penetration of a cone into an incompressible fluid is investigated. Formulas for velocity and deceleration during the penetration process are derived. The constants entering into the formulas may be obtained in the general case by numerical solution of an integral equation. Particularly examined is the case of penetration of a narrow cone for which the penetration law may be expressed by simple formulas. *Authors' summary.*

Gerber, Robert. Sur les solutions exactes des équations du mouvement avec surface libre d'un liquide pesant. J. Math. Pures Appl. (9) **34** (1955), 185–299.

A complete treatment is given, with proofs, of various results on gravity flows announced earlier [C. R. Acad. Sci. Paris **233** (1951), 1261–1263, 1560–1562; **235** (1952), 693–694, 1601–1602; MR **13**, 594; **14**, 168, 508]. The flows considered are assumed to be steady, two-dimensional, irrational and incompressible, and to take place in a canal with bottom S and free surface L. The first problem concerns flows in which S is periodic, symmetric and is monotonic between successive axes of symmetry, while L is assumed to increase and decrease with S. An existence theorem for such flows is proved under suitable limitations on the slope of S and on the size of the Froude number $\gamma\psi_0/V_0^3$, where γ is the acceleration of gravity, ψ_0 is the total flux and V_0 is the speed at a crest. The methods used are to reduce the problem to a system of non-linear integral equations, to obtain a priori bounds for the solutions and then after introducing appropriate Banach spaces, to apply the methods of Leray and Schauder.

After imposing additional restrictions on the Froude number, the author also succeeds in proving a uniqueness theorem for the above type of periodic flows, providing S has Hölderian curvature, but is not a horizontal straight line.

The next problem considered is that in which both the bottom S and the free line L are assumed to be non-increasing, and to have horizontal asymptotes at plus and minus infinity. Here V_0 is taken to be the limiting upstream speed on L. Under suitable restrictions on the slope of S, it is proved by again using the Leray-Schauder procedure, that this problem will always have at least one solution, providing that $\gamma\psi_0/V_0^3<1$. Finally, the question of the analyticity of the free line in such gravity flow problems is discussed, and a lemma is proved similar to that of H. Lewy [Proc. Amer. Math. Soc. **3** (1952), 111–113; MR **14**, 168]. *D. H. Hyers* (Los Angeles, Calif.).

Brillouët, Georges. Ondes liquides de gravité abordant une plage inclinée sur l'horizon de l'angle: $\alpha=\pi/2q$, (q entier). C. R. Acad. Sci. Paris **240** (1955), 1189–1191.

Brillouët, Georges. Ondes liquides de gravité abordant une plage inclinée sur l'horizon. C. R. Acad. Sci. Paris **240** (1955), 1310–1312.

These two papers extend the results of a previous note [same C.R. **239** (1954), 860–862; MR **16**, 296] in which the author considered gravity waves in the presence of a vertical cliff. He now considers the case of a sloping bottom, making an angle with the horizontal of $\alpha=\pi/2q$ and $\alpha=p\pi/2q$, respectively, in the two papers under review; here p and q are integers, p odd and prime to q, $1<p<2q$. Denote by $\mathcal{D}$ the wedge bounded by Ox and the bottom Ot, and let $f(z)$ be a complex potential (i.e. $\varphi(x, y)=\operatorname{Re} f(z)$ is the velocity potential). Then, with the assumptions customary for such problems (perfect fluid and linearized free-boundary condition) the problem is to determine a function $f(z)$ analytic in $\mathcal{D}$ and satisfying

$$\operatorname{Im}(f'+if)=0 \text{ on } Ox, \quad \operatorname{Im}(e^{-ip\pi/2q}f')=0 \text{ on } Ot,$$

where in the first paper $p=1$. Let $\varrho_j=ie^{(\pi p j/q}$ and

$$a_j=\operatorname*{res}_{\zeta=\varrho_j} q^{\lambda}\frac{\zeta^{q-1}}{(\zeta-\varrho_1)\cdots(\zeta-\varrho_q)}, \quad \lambda=(2n+1)q/p.$$

Then the general solution of the problem may be expressed as a linear combination of the functions

$$(*) \qquad f_0(z)=\sum_{j=1}^{q} a_j e^{\varrho_j z}, \quad f_{1,\lambda}(z)=\sum_{j=1}^{q} a_j e^{\varrho_j z} L_{j,\lambda}(z),$$

where $L_{j,\lambda}(z)=\int e^t t^{-\lambda}dt$ if $p\equiv 1 \pmod 4$, $=\int e^{-t}t^{-\lambda}dt$ if $p\equiv 3 \pmod 4$ and appropriate paths of integration, ending in $-\varrho_j z$ and $\varrho_j z$ respectively in the two cases, are taken.

For the case $p=1$ (first paper) f_0 is regular in $\mathcal{D}$ and bounded at infinity; $f_{1,\lambda}$ for $n<0$ is a polynomial in z, hence regular in $\mathcal{D}$, unbounded at infinity; $f_{1,0}$ has a logarithmic singularity at the origin, $f_{1,\lambda}$, $n>0$, a pole of order $2nq$; both are bounded at infinity.

For the case $p\neq 1$ (second paper) none of the solutions (*) is bounded at infinity; at the origin f_0 and $f_{1,\lambda}$, $\lambda<q$, are finite, $f_{1,q}$ has a logarithmic singularity and $f_{1,\lambda}$, $\lambda>q$,

behaves like $1/x^{\frac{1}{2}-q}$. However, by taking linear combinations of (*), there exist solutions, unique up to a multiplicative constant, bounded at the origin and infinity but not vanishing on the free surface at infinity, and similarly with a logarithmic singularity at the origin. Other similar results. No proofs are given. *J. V. Wehausen.*

Ursell, F. Water waves generated by oscillating bodies. Quart. J. Mech. Appl. Math. **7** (1954), 427–437.

This paper amplifies the author's results in two earlier papers [same J. **2** (1949), 218–231; Proc. Roy. Soc. London. Ser. A. **220** (1953), 90–103; MR **11**, 65; **15**, 571]. Although no new results are derived, he gives a different approach to the results of the second paper cited above which makes intuitively acceptable the results of that paper. He suggests that the same approach might be valuable for wave resistance of thin ships.

J. V. Wehausen (Providence, R.I.).

Havelock, Thomas. Waves due to a floating sphere making periodic heaving oscillations. Proc. Roy. Soc. London. Ser. A. **231** (1955), 1–7.

The author considers the wave motion resulting from forced vertical oscillation of a sphere half-immersed in a heavy fluid. The amplitude of oscillation is assumed small and the boundary condition at the free surface linearized. After deducing some general formulas for periodic singularities, the author reduces the problem to one of solving an infinite set of equations in infinitely many unknowns. Numerical calculations were carried out for several values of $\sigma^2 a/g$ (σ=frequency, a=radius) up to 3.0. Results for the virtual inertia coefficient and a "damping" coefficient are displayed graphically. Unlike the result obtained by Ursell [Quart. J. Mech. Appl. Math. **2** (1949), 218–231; MR **11**, 65] for the circular cylinder, the former approaches a finite limit, 0.828, as $\sigma^2 a/g \to 0$. *J. V. Wehausen* (Providence, R.I.).

Kostyukov, A. A. The velocity potential and wave resistance of ships with finite water depth. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 9, 50–62. (Russian)

The author finds the velocity potential, form of the free surface at a distance, and the wave resistance of a "thin" ship moving in water of finite depth. In order to find the potential function, the author expresses it as a Fourier transform with respect to the vertical variable. The results are applied to a moving submerged dipole and a submerged propeller (treated as a disc with constant density of sources). The results agree with known ones.

J. V. Wehausen (Providence, R.I.).

Laporte, Otto, and Yoshihara, Hideo. A rigorous method for finding the lift of a certain class of airfoils and remarks on the meaning of Schrenk's approximate rule. J. Aero. Sci. **22** (1955), 787–794.

The authors present a method for finding the lift of a family of twisted wings in incompressible flow. The midchord points are unswept and the chords themselves are given by

$$c(y)=c_0[1-(2y/b)^2]^{\frac{1}{2}}[1-t(2y/b)^2]^{\frac{1}{2}}$$

where y is the span coordinate, b is the span, c_0 is the maximum chord and t ($-1\leq t\leq 1$) is the taper parameter. The lift obtained is approximated in such a way as to derive Schrenk's approximation for untwisted wings [NACA Tech. Memo. no. **948** (1940)]. In this way the authors are able to justify the validity of Schrenk's rule and they generalize it to apply to twisted wings. The work is, in parts, difficult to follow owing to many misprints involving K, the complete elliptic integral, and k, the modulus of the elliptic functions. *G. N. Lance.*

★ **Leyds, J. Lift and moment of low aspect ratio wings in incompressible unsteady flow.** Proceedings of the Second Canadian Symposium on Aerodynamics, Toronto, 1954, pp. 221–237. The Institute of Aerophysics, University of Toronto, Toronto, 1954.

This paper is an extension of some work by Lawrence and Gerber [J. Aero. Sci. **19** (1952), 769–781]. It takes account of the fact that the lift and moment coefficients at points on a wing, in incompressible unsteady flow, are functions of position on the wing. The results, which contain the coordinates of the point considered, are considerably more complicated than those of Lawrence and Gerber. No comparison is given to show what improvement has been achieved in an actual flutter calculation.

G. N. Lance (Southampton).

Pasqualini, C. Lift distribution with nonlinear diagram. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. no. **206**, Serie Tercera. Publ. Esp. **43** (1953), 137–151. (Spanish. English summary)

The author considers a generalization of Prandtl's lifting-line integral equation, namely, the relations

$$8F(y)=\tau(y)\cdot L(\beta);\ \beta=\alpha(y)+\alpha_i(y);$$

$$\pi\alpha_i(y)=-\int_{-1}^{1}(y-x)^{-1}F'(x)dx$$

with y indicating a dimensionless coordinate over the span of a finite wing, while α means the angle of incidence and α_i the induced angle of incidence. The function $\tau(y)$ varies with the chord; $L(\beta)$ being proportional to β in Prandtl's classical equation, is now considered as some nonlinear function of β, and for practical purposes, a polynomial is chosen. The unknown function $F(y)$, describing the distribution of circulation, is set up as a power series $F=\sum_{n=1}^{\infty}\alpha^n F_n/n!$. The coefficients F_k are determined from integral equations which contain F_k in linear form and the preceding coefficients in a nonlinear composition. A numerical example for the trapezoidal wing and coefficient formulas for the elliptic wing is presented. *H. F. Bueckner* (Schenectady, N.Y.).

Hawthorne, W. R. Rotational flow through cascades. I. The components of vorticity. Quart. J. Mech. Appl. Math. **8** (1955), 266–279.

Starting with the equations of inviscid, incompressible flow with body forces, an expression is derived for the rate of change, in the direction of flow, of the vorticity component in the flow direction. This formula exhibits three terms, one of which is independent of the body forces; this is the one obtained by the author in an earlier investigation [Proc. Roy. Soc. London. Ser. A. **206** (1951), 374–387; MR **13**, 177] and by Squire and Winter in a small-perturbation analysis [J. Aero. Sci. **18** (1951), 271–277]. The other two terms are shown to represent, in the case of flow through a cascade of airfoils, (i) trailing vortices in the sense of Prandtl's wing theory and (ii) additional trailing vortices due to stretching and distortion of the original (upstream) vortex lines. The component (ii) was first noticed by Preston in an approximate treatment of the same case [Aero. Quart. **5** (1954), 218–

234]. The analysis is repeated by considering conservation, distortion, and stretching of vortex lines only. The reviewer remarks that these considerations do not alter the results of calculations like those of Ehrich [J. Aero. Sci. 22 (1955), 51–60] but do show that his implied identification of all the trailing-vortex strength behind the airfoils as Prandtl vortices was not generally correct.

W. R. Sears (Ithaca, N.Y.).

Hawthorne, W. R., and Armstrong, W. D. Rotational flow through cascades. II. The circulation about the cascade. Quart. J. Mech. Appl. Math. 8 (1955), 280–292.

The conclusions of Part I (reviewed above) are applied to a certain cascade representing impulse-turbine blading, and the results are compared with experiment. It is found that for this cascade the two parts of the trailing-vortex strength mentioned above are about equal. It is concluded also that circulation increments due to secondary flow can be satisfactorily estimated. *W. R. Sears.*

Kao, Shih-Kung. Wave motion in a rotating Couette flow of a viscous fluid. Tellus 7 (1955), 372–380.

A particular set of solutions to the equations of motion of an incompressible viscous fluid are obtained. The basic velocity is a Couette flow on a rotating earth where the variation of the Coriolis parameter with latitude is constant. The solutions represent waves with varying amplitude, velocity, and inclination of trough and ridge lines. Several properties of the flow are examined and it is shewn that all waves of this class are eventually damped out, even if viscosity is absent. Energy and momentum transports are computed and these enable the author to find the error introduced by the use of geostrophic velocities in momentum transport calculations.

M. H. Rogers (Urbana, Ill.).

Moisil, Gr. C. Une décomposition des mouvements plans, lents et non-permanents des liquides visqueux incompressibles. Com. Acad. R. P. Române 1 (1951), 233–237. (Romanian. Russian and French summaries)

The author establishes the result that the plane, slow and unsteady motion of a viscous incompressible fluid is a superposition of two movements, i.e. the irrotational motion, and the motion in which the mean pressure remains constant. The author's investigation is based on the linear hydrodynamic equations, from which the equation for the stream-function Ξ is obtained, while the mean-pressure $p+i(\mu\Delta\Xi-\varrho\cdot\partial\Xi/\partial t)$ is monogenic in $x+iy$ (p is harmonic). The equation for the stream-function is decomposed in the well-known manner, and the same process is applied to the velocity components and the various tensorial components. *K. Bhagwandin.*

Iacovache, M. Relations entre les tensions dans un liquide visqueux incompressible, en mouvement lent, permanent. Com. Acad. R. P. Române 1 (1951), 245–249. (Romanian. Russian and French summaries)

The author applies the matrix method of Gr. C. Moisil to obtain conditions of compatibility for the stresses in the theory of slow motions of a viscous incompressible fluid. They have the same form as St. Venant's conditions of compatibility for infinitesimal strain. [The reviewer regards it as obvious that for such fluids these relations hold in all motions, slow or not.] *C. A. Truesdell.*

Tipei, N. La solution générale des équations hydrodynamiques de la lubrification. Com. Acad. R. P. Romîne 4 (1954), 501–507. (Romanian. Russian and French summaries)

The author establishes the generalized pressure-equation in the hydrodynamical region, and states the conditions (sufficient in practice) for the solution of such an equation. Finite-difference methods are applied. As a particular case the author gives the expression for constant large surfaces. *K. Bhagwandin* (Oslo).

Filimon, I. Mouvements bidimensionnels dans un milieu poreux. Com. Acad. R. P. Române 2 (1952), 325–327. (Romanian. Russian and French summaries)

The author applies conformal mapping theory to study the two-dimensional problem of the infiltration of an incompressible fluid in a porous homogeneous medium. The author starts from the equation $\Delta p(x, y)=0$, subject to the boundary-conditions

$$p=\omega x+\delta \text{ for } 0\leqq x\leqq 1 \text{ when } y=0,$$

$$\chi\cdot\partial p/\partial x=0 \text{ for } x=0,\ x=l \text{ when } y>0.$$

The strip $0\leqq x\leqq l$ of the plane $z=x+iy$ is mapped unto the circle $|Z|\leqq 1$, by means of the relation

$$Z=(l/i\pi)\log[Z-i/Z+i]-l/2.$$

The solution found by the author is given by

$$p(x, y)=\delta+\tfrac{1}{2}\omega l-\frac{4\omega l}{\pi^2}\cdot\sum_{j=0}^{\infty}\frac{\cos(2j+1)^{\pi x/l}}{(2j+1)^2}\cdot e^{-(2j+1)\cdot\pi y/l}.$$

K. Bhagwandin (Oslo).

Carafoli, E., et Patraulea, N. Mouvement dans un milieu poreux autour des surfaces perméables. Com. Acad. R. P. Române 2 (1952), 143–146. (Romanian. Russian and French summaries)

The authors study the movement of an incompressible fluid past a permeable plate according to the Darcy-hypothesis. The penetration conditions through the permeable surface are stated. The authors find that when the intensity of the vortex layer is developed in a Fourier series, an equation is arrived at which is virtually the same as that encountered in the linear theory of wings of finite span [cf. Carafoli, Théories des ailes monoplanes d'envergure finie. An. Acad. Române. Mem. Secț. Ști. (1945) (unavailable for review)]. By means of a simple transformation the authors determine the movement in the whole plane. *K. Bhagwandin* (Oslo).

Patraulea, N. Une solution du type Oseen pour l'écoulement autour des surfaces perméables. Com. Acad. R. P. Române 2 (1952), 737–743. (Romanian. Russian and French summaries)

The author applies an Oseen-type approximation to the problem of the motion past permeable surfaces. Boundary-conditions are stated, and the solution is expressed in terms of an unsolved integral equation. *K. Bhagwandin.*

Gheorghiță, Șt. I. Sur certains mouvements en milieux poreux hétérogènes. Com. Acad. R. P. Romîne 5 (1955), 509–513. (Romanian. Russian and French summaries)

The author derives an integral equation for the plane movement of a fluid in a non-homogeneous porous medium. In the case of an ellipse (considered as a separation curve), the author obtains an analytical solution.

The author also draws a parallel between the present problem and an allied problem encountered in electrostatics [cf. G. Power, Amer. Math. Monthly **58** (1951), 249–253; MR **12**, 826]. *K. Bhagwandin* (Oslo).

Gheorghiță, Șt. I. Au sujet des mouvements des fluides incompressibles en présence des corps poreux. Com. Acad. R. P. Romîne **5** (1955), 661–663. (Romanian. Russian and French summaries)

The author propounds new boundary-value problems in the theory of the movement of incompressible fluids in the presence of bodies of weak porosity. The investigation is founded on complex-function theory. Power-series developments in terms of the filtration coefficient are given for the case of an cylinder of finite radius. A similar expression is given for the case of a sphere in a viscous fluid, generalizing an expression given by Stokes. Blasius-Čaplygin's formulas (for the plane stationary and irrotational movements of perfect fluids) are also generalized. *K. Bhagwandin* (Oslo).

Vasilache, S. Sur le mouvement d'un fluide incompressible dans un cylindre poreux semi-infini. Com. Acad. R. P. Române **1** (1951), 331–336. (Romanian, Russian and French summaries)

The author considers the steady motion of an incompressible fluid through a porous and isotropic cylinder of radius a and of infinite extent in the positive z-direction. The investigation is founded on the Darcy formula $\mathbf{v}=\chi \operatorname{grad} p$ ($\mathbf{v}$=velocity, χ=permeability-coefficient, p=pressure), and on a continuity equation [cf. L. S. Leibenzon, Yubileĭnyĭ sbornik, v. 2, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1947, pp. 608–619]. From the supposition $\chi=De^{-\beta z}$ the author obtains the equation

$$(*) \qquad \frac{\partial^2 p}{\partial z^2}+\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}-\beta\frac{\partial p}{\partial z}=0,$$

subject to the boundary-conditions

$$(**) \qquad p(r,0)=f(r), \text{ and } \partial p/\partial r=0, \text{ for } 0\leqq r\leqq a.$$

The final solution is given by

$$(***) \quad p(r,z)=\frac{2}{a^2}\left\{\int_0^a pf(r)dr+\sum_{n=1}^{\infty}e^{-k_n z}\frac{J_0(\alpha_n r)}{[J_0(\alpha_n a)]^2}\int_0^a sf(s)J_0(\alpha_n s)ds\right\},$$

where $\alpha_n^2=k_n^2-k_n\beta$ are the roots of the equation $J_1(\alpha a)=0$. *K. Bhagwandin* (Oslo).

Vasilache, S. Sur le mouvement d'un fluide incompressible dans un cylindre poreux d'une longueur finie. Com. Acad. R. P. Române **1** (1951), 551–555. (Romanian. Russian and French summaries)

This paper is a continuation of the preceding one, save for the additional boundary-condition $p(r,l)=g(r)$, to account for the finiteness of the length of the cylinder. The analysis is approximately the same, too. The expression for the pressure $p(r,z)$ is too complicated to be reproduced here. It is to be regretted that there are so many printing errors in the paper. *K. Bhagwandin.*

Dem'yanov, Yu. A. On an application of A. A. Dorodnicyn's variables in boundary-layer theory. Prikl. Mat. Meh. **19** (1955), 507–508. (Russian)

In steady two-dimensional compressible flow without pressure gradient the change of variables, $\xi=x$, $\eta=\int_0^y(\varrho/\varrho_\infty)\,dy$, ϱ_∞ the density outside the boundary layer, is used to express the boundary-layer equations in terms of a stream function and the enthalpy. The author then shows how particular assumptions can be used to obtain qualitative results in the general case. *L. M. Milne-Thomson* (Greenwich).

Dorfman, A. Š., and Švec, I. T. Some particular cases of solution of the boundary-layer equations for a compressible fluid. Prikl. Mat. Meh. **19** (1955), 509–512. (Russian)

See the preceding review. The boundary-layer equations are transformed by

$$\xi=\int_0^x(p/p_0)^n dx, \quad \eta=\int_0^y(\varrho/\varrho_0)dy,$$

where suffix zero refers to conditions outside the layer. Taking $\mu/\mu_0=T/T_0$ and putting Prandtl's number equal to unity greatly simplifies the equations for steady conditions with pressure gradient. The case $u=c\xi^m$ for the velocity just outside the boundary layer is then shown to lead to the same ordinary differential equation for the determining function as incompressible flow with the law $u=cx^m$. The values for this case have been tabulated [Kočin, Kibel', and Roze, Theoretical hydrodynamics, v. 2, Gostehizdat, 1948, § 34, Tables VII, VIII; for a German translation see MR **17**, 307]. The singular case $m=-1$ is also discussed. *L. M. Milne-Thomson* (Greenwich).

Eichelbrenner, E.-A., et Oudart, A. Méthode de calcul de la couche limite tridimensionnelle. Application à un corps fuselé incliné sur le vent. O.N.E.R.A. Publ. no. **76** (1955), 49 pp.

It is proposed to calculate three-dimensional boundary-layer flows by decomposing the velocity vector into components along and across the potential-flow streamlines; it can usually be assumed that the latter component is small. If so, it may be permissible to neglect this ("secondary") component in calculating the other ("primary") one, whose direction lies along the streamlines. This scheme is applied here to the case of an ellipsoid of revolution (prolate spheroid) at a small angle of incidence in a stream. The calculations are carried out by the Kármán-Pohlhausen scheme, applied to both components in turn. For the initial thickness of the "primary" boundary layer, reference, is made to studies of stagnation-point flow. In the absence of better information, the thickness of the "secondary" boundary layer is taken to be the same.

A numerical example is worked out, and the results are compared with experimental observations. In the tests, the flow of water about an ellipsoid was rendered visible by use of colored fluid. The agreement of test and theory, especially as regards the position of flow separation, is considered very satisfactory. *W. R. Sears.*

Massignon, Daniel. Grandeurs locales fines et grossières, fluctuations, dispersions et corrélations en hydrodynamique statistique. C. R. Acad. Sci. Paris **241** (1955), 1259–1262.

★**Kuroda, Yoshiteru. Note on the fourth-order moments of random velocities.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 219–221. Science Council of Japan, Tokyo, 1953.

Chuang, Feng-Kan. On the decay of turbulence. Chinese J. Phys. 9 (1953), 201–214. (Chinese. English summary)

This paper consists of two parts. The first part gives a description of the motion of large eddies in a turbulent flow. The non-stationary character of the large eddies is emphasized. Up to present, there appears to be some confusion regarding the law of turbulence decay, especially the variation of the microscale with time. This paper introduces a new characteristic length for large eddies which leads to a new decay law valid at the initial period. The apparent discrepancies between Kolmogoroff's decay law and Lin's decay law are seen to be due to different expansions of the present one. It is hoped that the physical picture described herein would give some further insight into the structure of turbulence. In the second part, an analysis similar to that adopted by Sedov for the correlation coefficients is applied to the turbulent spectrum. New results are obtained, in particular the transition of the decay law from the initial period to the final period.

Autoreferat (Zbl **52**, 215).

Chisnell, R. F. The normal motion of a shock wave through a non-uniform one-dimensional medium. Proc. Roy. Soc. London. Ser. A. **232** (1955), 350–370.

Une onde de choc plane perpendiculaire à Ox, se propage avec une vitesse constante dans un fluide au repos à pression constante; la masse spécifique ϱ avant le choc est constante à l'exception de la région $0<x<h$ dans laquelle on a $\varrho=\varrho(x)$, la fonction $\varrho(x)$ etant croissante. L'auteur détermine la trajectoire du choc dans la région de densité variable. La fonction $\varrho(x)$ est supposée constante par intervalles; les diverses valeurs étant infiniment voisines. En négligeant les ondes réfléchies après le choc l'auteur en déduit sous la forme $\varrho=Af(z)$, une relation approchée entre la masse spécifique et l'intensité z du choc; A est une constante et z le rapport des pression après et avant le choc; la fonction $f(z)$ est explicitée et ses valeurs numériques calculées. En tenant compte ensuite des ondes réfléchies, l'auteur établit une seconde formule relation plus précise sous la forme $\varrho=Ag(z)$. *H. Cabannes.*

Linnaluoto, V. V. A numerical integration method for calculating the pressure distribution at supersonic speeds for wings with subsonic leading edge at symmetrical flow conditions. Svenska Aeroplan A. B. Tech. Note No. 6 (1952), 19 pp.

In der linearisierten Überschalltragflächentheorie kann oftmals die Ermittlung sowohl des Störgeschwindigkeitspotentials als auch seiner Ableitungen durch Belegung des Flügeleinflußgebiets mit quellartigen Überschallsingularitäten bewerkstelligt werden. In allen Fällen, in denen die Quellstärke direkt durch die Randbedingungen vorgegeben ist, hat man es alsdann — wenn ohne Beschränkung der Allgemeinheit die Machzahl $M=\sqrt{2}$ gewählt wird und wenn man sich nur für die genannten Größen auf der Flügelfläche selbst interessiert — nur mit der Berechnung von Doppelintegralen der Gestalt

$$\iint_{(G)} \frac{\lambda(\xi,\eta)d\xi d\eta}{[(x-\xi)^2-(y-\eta)^2]^{\frac{1}{2}}}$$

zu tun. Das Integrationsgebiet G ist dabei das im Vorkegel von x, y liegende Quellgebiet der ξ, η-Ebene. Für den Fall beschränkter $\lambda(\xi,\eta)$ gibt Verf. ein für ingenieurtechnische Zwecke geeignetes Verfahren zur approximativen Berechnung solcher Integrale an. Es besteht darin, daß: 1) statt ξ, η (bzw. x, y) charakteristische Koordinaten $\sigma=(\xi-\eta)/\sqrt{2}$, $\tau=(\xi+\eta)/\sqrt{2}$ (bzw. entsprechend s, t) eingeführt werden; 2) das entstehende Doppelintegral $\iint\lambda(\sigma,\tau)[(s-\sigma)(t-\tau)]^{-\frac{1}{2}}d\sigma d\tau$ mittels einer quadratischen Gitteraufteilung der ξ, η-Ebene (parallel den σ, τ-Achsen) näherungsweise nach dem Mittelwertsatz der Integralrechnung als

$$\sum_{\mu,\nu}\lambda(s_\mu,t_\nu)\iint_{Q_{\mu,\nu}}\frac{d\sigma d\tau}{[(s_\mu-\sigma)(t_\nu-\tau)]^{\frac{1}{2}}}$$

[$\lambda(s_\mu,t_\nu)$ Mittelwert von λ im Quadrant $Q_{\mu,\nu}$] geschrieben wird; 3) die $\iint_{Q_{\mu,\nu}}$ ein für allemal berechnet und als „Matrix der Abstandsfunktion" auf durchscheinendes Papier aufgetragen werden. Im gleichen Maßstab wird 4) die „Matrix der Belegungsfunktion" $\lambda(s_\mu,t_\nu)$ in den Flügelgrundriß eingeschrieben und 5) dann die durchsichtige Abstandsmatrix auf die so präparierte Flügelzeichnung gelegt, wonach sich leicht die Doppelsumme reihenweise berechnen läßt. Gesonderte Formeln bzw. Kurven geben Hilfswerte zur Berechnung der Beiträge der von den Flügelrändern angeschnittenen Quadrate der Abstandsmatrix. — Es sei erwähnt, daß der Titel der Note insofern irreführend ist, als das Verfahren keineswegs auf den darin genannten Spezialfall beschränkt ist.

H. Behrbohm (Zbl **49**, 135).

★ **Etkin, B., and Woodward, F. A. Lift distribution on supersonic wings with subsonic leading edges and arbitrary angle of attack distribution.** Proceedings of the Second Canadian Symposium on Aerodynamics, Toronto, 1954, pp. 80–105. The Institute of Aerophysics, University of Toronto, Toronto, 1954.

A numerical method is given for the calculation of the lift distribution on twisted and cambered wings in supersonic flow. The leading edges, which may be either straight or curved are assumed to be subsonic. Two examples are presented and the results so obtained are compared with exact (linearized theory) results, the agreement is good. In the case of curved leading edges exact solutions have not been obtained and one is forced to use numerical techniques; however, for straight leading edges numerical methods are less desirable since exact solutions exist for many cases of twist and camber [Lance, Aero. Quart. 6 (1955), 149–163; MR **16**, 971]. The method is suitable for use on a high-speed digital computer. *G. N. Lance* (Southampton).

★ **Kusukawa, Ken-ichi. On the supersonic flow of elastic medium past a wedge.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 5–9. Science Council of Japan, Tokyo, 1952.

★ **Miles, John W. Unsteady supersonic flow.** Air Research and Development Command, Baltimore, Md., 1955. xii+569 pp.

This monograph, by an acknowledged authority in the field of unsteady aerodynamics, is an up-to-date account of unsteady flow, written primarily for the theoretical aerodynamicist. The word supersonic, which appears in the title, is purposely omitted here since subsonic flow is also covered in sufficient detail to show the relationship between the two types of unsteady flow. The mathematics required for a complete understanding of the work may be termed "advanced". Fourier-transform theory is used extensively; this in one reason why such a vast amount of material can be condensed into a comparatively small space. The mathematics is rigorous throughout; in some cases proofs are omitted, but complete references are

invariably given. The monograph is made all the more readable by the fact that cross references are numerous. An index of authors is presented at the end.

There are fourteen chapters and an appendix. The first two chapters are concerned with a derivation of the linearized equations of motion and a discussion of subsonic phenomena. The various conditions which must be satisfied by the parameters: aspect ratio, Mach number, reduced frequency and fineness ratio in order that the differential equation for the velocity potential should take special forms are very clearly listed in a Table. Chapters 3 and 4 deal with the transformations which can be applied to the equation satisfied by the velocity potential and with the solution of this equation by separation of variables. In Chapter 5 it is shown how standard steady-flow results can be used to solve unsteady problems. Chapters 6 through 11 contain a discussion of slender-wing theory, including the rectangular, quadrilateral and delta wings, both with subsonic and supersonic leading edges. Chapter 12 deals with slender bodies and wing-body combinations. The problem of accelerated flight is treated in Chapter 13 and a discussion of accelerated flight through the transonic regime is included. A few non-linear problems, which are amenable to calculation, are treated in the last chapter. The appendix deals with reversed-flow theorems, and much of this work is presented here for the first time.

The reviewer fears that the fact that this monograph is distributed by the Air Research and Development Command of the U.S. Air Force will greatly reduce its circulation. If this fear is justified, it is very regrettable, since this is the first time so much work on unsteady flow has been presented in one volume. *G. N. Lance.*

Lin, T. C., and Street, R. E. Effect of variable viscosity and thermal conductivity on high-speed slip flow between concentric cylinders. NACA Rep. no. **1175** (1954), ii+36 pp. (1955).

The problem described by the title is considered in great detail starting from the Burnett equations with coefficients of heat conductivity and viscosity dependent upon the temperature. The paper is an extension of work done by Schamberg [thesis, Calif. Inst. Tech., 1947] who solved the corresponding problem with constant coefficients. The method of solution is an iterative one based upon obtaining a lowest approximation with constant coefficients and using the resulting temperatures to find the approximate variable coefficients of viscosity. The results are expressed in terms of approximately defined "effective coefficients." From very complicated equations, numerous graphs are obtained using values of the various parameters representative of the properties of air with viscosity proportional to the temperature. *G. Newell.*

★**Skudrzyk, Eugen. Die Grundlagen der Akustik.** Springer-Verlag, Wien, 1954. xxii+1084 pp. $ 35.00.

This is a very substantial treatise. The author announces as his purpose an exposition of the foundations which are essential for all acoustical investigations. In this he desires to stress the manifold phenomena which make acoustics a significant and fascinating branch of modern physics as well as a very useful one to technologists through its increasingly important applications in all phases of human activity. The aim has been achieved in comprehensive fashion: the thirty-six chapters cover a vast amount of acoustics and few topics of importance are omitted.

Inevitably most of the treatment follows conventional lines. However, certain topics come in for more than routine attention and many of these are of mathematical interest. One example is the detailed examination of pulse transmission through linear mechanical systems subject to frequency and other limitations, leading to effective use of Fourier and Laplace transform techniques, which are carefully explained. Another illustration is the thorough study of diffraction and scattering problems involving the solution of the wave equation subject to practical boundary conditions. There is heavy emphasis on the use of electro-mechanical analogies, which will please the acoustical engineer. The physicist interested in the application of ultrasonics to the study of the structure of matter will find a long chapter devoted to the absorption of sound in fluids and solids with ample justice given to relaxation mechanisms.

Each chapter is prefaced by a brief but helpful summary in non-analytical form with emphasis on the physical significance and purpose of the treatment to follow. A very extensive bibliography is included at the end of the volume. Its utility is diminished, however, by the general absence of specific references in the body of the text. The illustrations, which are practically all line drawings, are very clear and well reproduced.

The reviewer has been able to detect few errors of consequence, though the treatment of group velocity on page 143 is so sketchy as to be misleading.

This book can be recommended without reservation as a very useful reference work to all who are interested in acoustics. *R. B. Lindsay* (Providence, R.I.).

See also: Yanenko, p. 371; Synge, p. 373; Langefors, p. 412; Clark, p. 413; Gustafson, p. 444.

Elasticity, Plasticity

Adkins, J. E., and Rivlin, R. S. Large elastic deformations of isotropic materials. X. Reinforcement by inextensible cords. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1955), 201–223.

This paper deals with some simple large deformations of incompressible, isotropic, perfectly elastic bodies reinforced by thin, flexible and inextensible cords lying parallel in layers. These layers are regarded as boundary surfaces which divide the body into sections and which are inextensible in certain directions. The sections thus obtained are thought of as unreinforced elastic bodies subject to constraints on some of their boundaries. Classes of simple deformations of such bodies, possible for any form of the strain-energy function in the absence of body forces, were determined in the previous work of the authors [Adkins and Rivlin, same Trans. **244** (1952), 505–531; Rivlin, **242** (1949), 173–195; Proc. Roy. Soc. London. Ser. A. **195** (1949), 463–473; MR **14**, 221; **11**, 626; **10**, 650].

The constraints at the boundary impose certain relations between the parameters specifying those general deformations. Considering deformations of the sections in this way and fitting the sections together again after deformation, the following simple deformations for some reinforced bodies are obtained: 1) Pure homogeneous strain of a thin uniform sheet with a double layer of cords in the middle. 2) Combined pure homogeneous strain and flexure of a cuboid with a double layer in a plane

parallel to a pair of opposite faces. 3) Combined extension and flexure of a thin rectangular sheet with a plane double layer. 4) Simultaneous extension, inflation and flexure of a cylindrical tube with one or two single layers of cords lying in helical paths concentric with the cylinder.

In all cases, relations are given between the tensions in the cords, the applied forces and the parameters of deformation. *W. Noll* (Los Angeles, Calif.).

Truesdell, C. Correction to: "Hypo-elasticity". J. Rational Mech. Anal. **4** (1955), 1019–1020.

See same J. **4** (1955), 83–133; MR **16**, 880.

Ericksen, J. L. Inversion of a perfectly elastic spherical shell. Z. Angew. Math. Mech. **35** (1955), 382–385. (German, French and Russian summaries)

The problem of inversion of a spherical shell of incompressible, perfectly elastic material is solved for a general strain-energy function Σ. The surfaces of the shell are shown to be free from applied tractions if

$$1) \qquad \int_{R_1}^{R_2} R^3(d\Sigma/dR)\, dR=0,$$

where the undeformed solid is bounded by the spheres $R=R_1$, $R=R_2$. [The symmetrical expansion of a thick spherical shell of incompressible, perfectly elastic material was solved for a general strain-energy function by A. E. Green and R. T. Shield, Proc. Roy. Soc. London. Ser. A. **202** (1950), 407–419; MR **12**, 218. A slight adaptation of their work yields the result (1). The condition (1) has also been obtained by A. E. Green for a compressible material, ibid. **227** (1955), 271–278; MR **16**, 764.] If the strain energy is of the form assumed in the Mooney-Rivlin theory for rubber, the author shows that the condition (1) may be satisfied. *A. E. Green* (Providence, R.T.).

Manacorda, Tristano. Sulla torsione di un cilindro circolare omogeneo e isotropo nella teoria delle deformazioni finite di solidi elastici incomprimibili. Boll. Un. Mat. Ital. (3) **10** (1955), 177–189.

Taking up the known general solution for finite elastic torsion and extension of an incompressible circular cylinder, the author specializes it to the case of a certain strain energy depending on three constants and including as a special case the Mooney-Rivlin form. He studies the equation asserting that the resultant traction on the plane ends vanishes. He shows that for the special type of strain energy he considers, corresponding to any given twist there is one and only one extension, and this extension is always positive; conversely, given any positive extension, there exists one and only one angle of twist. [This work is to be compared with recent approaches via perturbation schemes approximating to a solution in which the normal tractions on the ends are everywhere zero.] *C. A. Truesdell* (Bloomington, Ind.).

Maliţa, Mircea. Sur les équations de mouvement des corps élastiques à isotropie transversale. Com. Acad. R. P. Române **2** (1952), 681–689. (Romanian. Russian and French summaries)

The author applies the matrix method of Gr. C. Moisil so as to find stress and displacement functions for bodies with transverse isotropy in the linear theory of small elastic vibrations. *C. A. Truesdell.*

Manacorda, T. Sulla più generale teoria linearizzata delle trasformazioni reversibili adiabatiche. Riv. Mat. Univ. Parma **5** (1954), 233–253.

The paper extends Signorini's work on isothermal deformations of an elastic material to adiabatic deformations. The dilatation is assumed to depend on temperature only. After developing a general theory from energy considerations, the author uses it to treat the bending of a cylinder and the propagation of elastic waves. The reader is assumed to be familiar not only with Signorini's work but also with his notation. *D. R. Bland* (London).

Ionescu-Cazimir, V. Sur les équations de l'équilibre thermo-élastique. II. Les relations entre les tensions et la température. Com. Acad. R. P. Române **1** (1951), 171–177. (Romanian. Russian and French summaries)

[For part I see Acad. R.P. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2** (1950), 589–595; MR **13**, 600.] The author writes out some formulae for linear thermoelasticity. She also notes the extension of the Beltrami-Michell equations to this case. *C. A. Truesdell* (Bloomington, Ind.).

Ionescu-Cazimir, V. Sur les équations de l'équilibre thermo-élastique. III. Les relations entre les tensions. Com. Acad. R. P. Române **1** (1951), 385–390. (Romanian. Russian and French summaries)

For the problem of small thermoelastic oscillations, the author obtains a set of partial differential equations of fourth order satisfied by the stresses alone. These might have been obtained by straightforward elimination of the derivatives of the temperature which occur in the generalized Beltrami-Michell equations noticed in the author's previous paper. *C. A. Truesdell* (Bloomington, Ind.).

Vâlcovici, V. Sur les relations entre les tensions. Com. Acad. R. P. Române **1** (1951), 337–339. (Romanian. Russian and French summaries)

The author notes some obvious and, in the reviewer's opinion, well known forms of the conditions of integrability for linear elasticity and linear viscosity. For example, if $\sigma_{ij}'=\sigma_{ij}-\sigma_{kk}\delta_{ij}/(m+1)$, where σ_{ij} is the stress tensor and m is Poisson's ratio, then the σ_{ij}' satisfy equations of the same form as the conditions of compatibility, rather than the usual Beltrami-Michell equations. *C. A. Truesdell* (Bloomington, Ind.).

Kröner, Ekkehart. Der fundamentale Zusammenhang zwischen Versetzungsdichte und Spannungsfunktionen. Z. Physik **142** (1955), 463–475.

Wie von dem Verfasser in einer vorangehenden Arbeit [Z. Angew. Phys. **7** (1955), 249–257] gezeigt wurde, bestehen weitgehende Analogien zwischen den von elektrischen Strömen verursachten Magnetfeldern und den von Versetzungslinien im umgebendem Medium verursachten Spannungsfeldern. Analog zur Stromdichte $\mathfrak{j}=d\mathfrak{i}/d\mathfrak{f}$ wird hier eine Versetzungsdichte α mit Hilfe der Formel $\alpha=d\mathfrak{b}/d\mathfrak{f}$, wo $\mathfrak{b}$ den Burgers-Vektor einer Versetzung bedeutet, definiert. Ähnlich zur Gleichung $\operatorname{div}\mathfrak{j}=0$, folgt dann hier $\operatorname{Div}\alpha=0$. Naheliegend ist nun zu versuchen, ob nicht dem bekannten Zusammenhang $\operatorname{rot}\mathfrak{H}=(4\pi/c)\mathfrak{j}$ hier $\operatorname{Rot}\varepsilon=\alpha$ entspricht, wo ε den symmetrischen Verzerrungstensor bedeutet. In der zitierten Arbeit hat der Verfasser ε in die Teile $\varepsilon_1=\frac{1}{2}(\nabla\mathfrak{s}+\mathfrak{s}\nabla)$, der den gewöhnlichen Deformationstensor der Elastizitätstheorie entspricht und den Inkompatibilitätstensor $\varepsilon_2=\nabla\times\varepsilon\times\nabla$, der nur innerhalb der Versetzungslinien nicht gleich Null ist und in der gewöhnlichen Elastizitätstheorie wegen den Saint-Venantschen Kompatibilitätsbedingungen identisch verschwindet, zerlegt. Weiter wird jedoch gezeigt, dass weder $\operatorname{Rot}\varepsilon=\alpha$ noch $\operatorname{Rot}\varepsilon_2=\alpha$ richtig sein können. Man

muss deshalb auch nichtsymmetrische Versetzungstensoren ε zulassen, die bereits von W. Voigt [Lehrbuch der Kristallphysik, Leipzig-Berlin, Teubner, 1910, S. 596] untersucht wurden. Dabei wird die wichtige Erkenntnis gewonnen, dass der antimetrische Teil ε_a des Verzerrungstensors gerade die Gitterrotationen (bzw. Gitterkrümmungen) die infolge der Anwesenheit von Versetzungen (z.B. Korngrenzen) auftreten, beschreibt. Mit Hilfe dieses Resultates erhält der Verfasser die endgültige Formel

$$\text{(1)} \qquad \text{Rot}\,(\varepsilon_s+\varepsilon_a)=\alpha.$$

Weiter wird dann das Problem der Integration von (1) besprochen. Zu diesem Zwecke wird der nichtsymmetrische Spannungsfunktionstensor φ mit Hilfe der Zusammenhänge $\sigma=\text{Rot}\,\varphi$, $\text{Div}\,\varphi=0$ eingeführt (σ ist der Spannungstensor). φ kann als ein tensorielles Potential für Versetzungen gedeutet werden. *T. Neugebauer.*

Šemyakin, E. I. Lamb's problem for a medium with an elastic after-effect. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 193–196. (Russian)

For a medium of this type the stresses are taken in the form

$$\sigma_{ss}(t)=\lambda\theta(t)+2\mu\frac{\partial w(t)}{\partial z}-\int_0^t\left[\lambda'\theta(\tau)+2\mu'\frac{\partial w(\tau)}{\partial z}\right]h(t,\tau)d\tau,\ \cdots,$$

where λ', μ' are two new constants, $h(t,\tau)\equiv h(t-\tau)$, the function $h(t)$ is positive and decreasing as $t\to\infty$ and $\mu'\int_0^\infty h(\tau)d\tau<\mu$. Assuming $\lambda'/\lambda=\mu'/\mu\equiv\omega$, the potentials φ, ψ_i must satisfy the integrodifferential equations of the form

$$a^2\frac{\partial^2\varphi}{\partial t^2}=\Delta\varphi(t)-\omega\int_0^t h(t-\tau)\cdot\Delta\varphi(\tau)d\tau,\ \cdots.$$

Using further restrictions concerning the source and the function $h(t)$, the author infers that the solution of the problem can be obtained in the form

$$\tilde{u}(r,\theta,z,t)=\int_0^\infty\tilde{u}_0(r,\theta,z,\tau)R(t,\tau)d\tau,$$

where $\tilde{u}_0$ is the solution of Lamb's problem for an elastic half-space and $R(t,\tau)$ the solution of the one-dimensional problem:

$$\frac{\partial^2R(t,\tau)}{\partial\tau^2}-\omega\int_0^t h(t-t_1)\frac{\partial^2R(t_1,\tau)}{\partial\tau^2}dt_1=\frac{\partial^2R(t,\tau)}{\partial t^2},$$

$$R(t,\tau)=\partial R(t,\tau)/\partial t=0 \text{ at } t=0,$$

$$\left[R(t,\tau)-\omega\int_0^\tau(t-t_1)R(t_1,\tau)dt_1\right]_{\tau=0}=a(t).$$

W. S. Jardetzky (New York, N.Y.).

Moisil, Gr. C. Observations sur les équations de l'élasticité plane. Com. Acad. R. P. Române **1** (1951), 395–398. (Romanian. Russian and French summaries)

In plane elasticity, let the displacement vector be expressed in terms of potentials in the usual way [cf. § 130 of Love, Elasticity, 4th ed., Cambridge, 1927, for the generalization to three dimensions]. A formulation in complex variables, familiar for the case of equilibrium, is immediate. *C. A. Truesdell* (Bloomington, Ind.).

Cattaneo, Carlo. Compressione e torsione nel contatto tra corpi elastici di forma qualunque. Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 23–43.

When two elastic bodies of identical homogeneous and isotropic material meeting in a point O are subject to a normal force P, there arises a contact area, in first approximation an ellipse with axes a and b, where the normal forces are distributed according to known laws. When there is a sensible friction between the bodies there arise in addition tangential forces. The object of this note is the determination of these forces in the general case $a\neq b$. The case $a=b$ has been treated by J. L. Lubkin [J. Appl. Mech. **18** (1951), 183–187; MR **12**, 878] and by the author [Ann. Scuola Norm. Sup. Pisa (3) **5** (1953), 1–16; MR **14**, 701]. Using extensively the results published in a paper by M. Pacelli [ibid. (3) **9** (1955), 1–22; MR **17**, 358], the author calculates these forces and the resulting torsional moment under various hypotheses.

H. Bremekamp (Delft).

★ **Edamoto, Isao. An electrical method for solving the torsion problem of a cylindrical body.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 215–218. Science Council of Japan, Tokyo, 1952.

Berger, H. M. A new approach to the analysis of large deflections of plates. J. Appl. Mech. **22** (1955), 465–472.

Simplified nonlinear equations for a flat plate with large deflections are derived by assuming that the strain energy due to the second invariant of the middle-surface strains can be neglected. These equations are used to compute the deflection of uniformly loaded circular and rectangular plates with various boundary conditions. The deflections found by this method are used to obtain stresses for the circular plate. In all cases where comparisons could be made, the deflections and stresses agree with exact solutions within the accuracy required for engineering purposes. The physical justification for the approximations is not discussed. *A. E. Green.*

★ **Radok, Jens Rainer Maria. Die Stabilität der versteiften Platten und Schalen.** Dissertation, Technische Hochschule München, 1955. P. Noordhoff N.V., Groningen-Djakarta, 1955. 47 pp. $ 0.75.

Die vorliegende Behandlung der Stabilitätsprobleme der ebenen, rechteckigen Platten und der Kreiszylinderschalen mit Versteifungen ohne Drillsteifigkeit baut auf gewisse Lösungen auf, die ihrem Charakter nach als Quellenlösungen bezeichnet werden können. Es wird gezeigt, dass man diese singulären Lösungen durch Anwendung eines Grenzprozesses aus bestimmten stetigen Lösungen ableiten kann, und dass sie vollkommen den Spannungszustand in versteiften Wänden beschreiben.

Im Gegensatz zu der Energiemethode führt diese Deformationsmethode direkt zu endlichen, homogenen Gleichungssystemen, deren Determinanten die charakteristischen Gleichungen zur Bestimmung der kritischen Lasten liefern, vorausgesetzt, dass man es nur mit Versteifungen einer Art zu tun hat, d.h. Längs- oder Querversteifungen im Falle der Platten, Ring- oder Längsversteifungen im Falle der Schalen.

Wenn beide Arten von Versteifungen auftreten, ist man im allgemeinen berechtigt, die Längsversteifungen zu verteilen und sich mit orthotropen Wänden zu befassen. Dieser Fall ist gleichfalls in dieser Arbeit behandelt.

Erfüllen die Versteifungen weiterhin die Grundannahme dieser Arbeit, dass sie symmetrisch mit Bezug auf die Wandmittelebene sind, so ist die Ordnung der Determi-

nantengleichung für die Platten gleich der Anzahl der Versteifungen, für die allgemeinen Schalen dreimal und für die orthotropen Schalen der oben behandelten vereinfachten Theorie zweimal so gross.

Die Anwendung der vorliegenden Resultate für die Platten auf praktische Probleme bereitet wenig Schwierigkeiten. Als Beispiel ist das Problem der rechteckigen Platten mit einer zentralen Querversteifung eingehend behandelt und die Resultate sind in der Form von Tabellen und Kurvenscharen gegeben, mit deren Hilfe die Bestimmung der kritischen Lasten mittels eines Rechenschiebers erfolgen kann. Ein beigefügtes Zahlenbeispiel zeigt, dass Timoshenkos Grenzwerte für das Verhältnis der Steifigkeiten der Versteifungen und der Platten, bei denen gleichzeitiges Ausbeulen der Versteifungen und Platten („general instability") aufhört, von der vorliegenden Theorie nicht bestätigt werden. In der Tat verursacht die Gegenwart der Versteifung für bestimmte Seitenverhältnisse der Platte eine Einschnürung der Ausbeulform, die bei konstanter Weite und anwachsender Länge der Platte ausgeprägter wird. Ein Vergleich der Gewichte der unversteiften Wand, der versteiften Wand und der unversteiften Wand, die die gleiche kritische Last wie die versteifte hat, zeigt, dass von einem gewissen Seitenverhältnis an die versteifte Wand zu Gewichtsersparnis führt.

Im allgemeinen wird die Anwendung der oben entwickelten Theorie für die Schalen Näherungsmethoden erfordern. Einer der wenigen Fälle, für die dieses nicht nötig ist — axialsymmetrisches Ausbeulen eines Zylinders mit einer zentralen Ringversteifung — ist ohne Zahlenbeispiel gelöst.

Die weitere Ausdehnung der Gedankengänge dieser Arbeit auf symmetrische und unsymmetrische Versteifungen mit Drill- und Biegesteifigkeit dürfte keinen neuen Schwierigkeiten begegnen. Einführung von zwei sehr steifen Spanten, die nur in kleiner Entfernung von gegenüberliegenden Seiten liegen, sollte es ermöglichen, die Stabilität homogener oder in derselben Richtung versteiften Platten mit zwei zu einem beliebigen Grade eingespannten Seiten zu untersuchen. Zusätzliche äussere Belastungen — zum Beispiel, innerer Druck im Falle der Schalen — sind gleichfalls leicht in Betracht zu ziehen.

Im Falle von Versteifungen, die durch Nieten mit der Wand verbunden sind, mag es wegen der Elastizität der Nieten nötig sein, Befestigungskonstanten in die „inneren" Randbedingungen einzuführen. Diese Konstanten würden dazu dienen, den Nieten elastische Formänderung zu erlauben. *R. Gran Olsson* (Trondheim).

Pailloux, Henri. Equilibre des poutres droites. C. R. Acad. Sci. Paris 241 (1955), 1199–1200.

Chien, Wei-Zang. Continuous beams with non-uniform stiffness. Chinese J. Phys. 9 (1953), 170–182. (Chinese. English summary)

The method of continued fractions is used for the determination of the bending moments of a continuous beam with non-uniform stiffness. It is required to find the solution of the following set of three-moment equations together with a set of end conditions: (1) $M_x = p_x M_{x-1} + q_x M_{x-2} + F_x$ $(x=2, 3, \cdots, n)$. Here M_x is the moment at the support no. x, p_x and q_x are quantities denoting the relative stiffness, and F_x is the loading factor. Let α_x and β_x be two independent solutions of the homogeneous equation (2): $M_x = p_x M_{x-1} + q_x M_{x-2}$. Then β_x/α_x is the x-th convergent of the continued fraction

$$\frac{q_2|}{|p_2}+\frac{q_3|}{|p_3}+\cdots,$$

and we have recurrence formulae $\alpha_x = p_x\alpha_{x-1}+q_x\alpha_{x-2}$, $\beta_x = p_x\beta_{x-1}+q_x\beta_{x-2}$ with $\alpha_0=\beta_1=0$, $\alpha_1=\beta_0=1$. Hence the general solution of (2) is $M_x = P\alpha_x + Q\beta_x$ $(x=0, 1, \cdots, n)$, where P and Q are two integration constants. For (1) we have the following particular solution

$$M_x^*: M_0^*=0,\ M_x^* = \sum_{i=1}^{x} \frac{F_{i+1}}{q_{i+1}} \frac{\beta_i\alpha_x - \alpha_i\beta_x}{\beta_x\alpha_{i-1} - \alpha_i\beta_{i-1}} \quad (x=1, 2, \cdots, n).$$

Hence the general solution of (1) is given by $M_0=Q$, $M_x = P\alpha_x + Q\beta_x + M_x^*$ $(x=1, 2, \cdots, n)$. A straight-forward numerical method of solution based upon this method is given at the end of the paper. *Autoreferat* (Zbl 52, 205).

Hu, Hai-Chang. On the matrix theory of continuous beams on elastic foundation. Chinese J. Phys. 9 (1953), 183–191. (Chinese. English summary)

In this paper, we employ the theory of matrices and continued fractions for the solution of bending problem of continuous beams on elastic foundation with unyielding supports. End moments are obtained in explicit expressions. Accurate numerical results may be calculated from these expressions directly without solving simultaneous equations. *Autoreferat* (Zbl 52, 205).

★ **Muki, Rokuro. On the axisymmetric problem of elasticity theory for a region bounded by two spheroidal surfaces.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 135–140. Science Council of Japan, Tokyo, 1955.

The author considers the problem described in the title on the assumption that the surface tractions are prescribed. With the aid of the Boussinesq stress functions a solution in series form is sought; the problem is reduced to the solution of an infinite system of linear algebraic equations for the unknown coefficients of superposition. No specific applications are included. *E. Sternberg.*

Hu, Hai-Chang. On the three-dimensional problems of the theory of elasticity of a transversely isotropic body. Chinese J. Phys. 9 (1953), 130–147. (Chinese. English summary)

Consider a transversely isotropic elastic body in equilibrium. Take a rectangular coordinate system (x, y, z) with the axis z perpendicular to the plane of isotropy of the body. In this system of coordinates, the equation of equilibrium in terms of displacement (u, v, w) is

$$(1)\quad (B_{11}\partial^2/\partial x^2 + B_{66}\partial^2/\partial y^2 + B_{44}\partial^2/\partial z^2)u + B_{12}\partial^2 v/\partial x\,\partial y + B_{13}\partial^2 w/\partial x\,\partial z = 0,$$

$$B_{12}\partial^2 u/\partial x\,\partial y + (B_{66}\partial^2/\partial x^2 + B_{11}\partial^2/\partial y^2 + B_{44}\partial^2/\partial z^2)v + B_{13}\partial^2 w/\partial y\,\partial z = 0,$$

$$B_{13}\partial^2 u/\partial x\,\partial z + B_{13}\partial^2 v/\partial y\,\partial z + (B_{44}\partial^2/\partial x^2 + B_{44}\partial^2/\partial y^2 + B_{33}\partial^2/\partial z^2)w = 0,$$

where $B_{66}=B_{11}-B_{12}$ and $B_{11}, \cdots, B_{33}$ are constants depending on the elastic property of the body. A particular solution of (1) may be found by expressing u, v, w in terms of a function F:

$$u = -\frac{\partial^2 F}{\partial x\,\partial z},\quad v = -\frac{\partial^2 F}{\partial y\,\partial z},$$

$$w = (B_{11}B_{13}^{-1})\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + (B_{44}B^{-1}{}_{11})\frac{\partial^2 F}{\partial z^2}\right],$$

where F satisfies the equation

$$(2)\quad \left[B_{11}B_{44}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2+(B_{44}^2+B_{11}B_{33}-B_{13}^2)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2}{\partial z^2}+B_{33}B_{44}\frac{\partial^4}{\partial z^4}\right]F=0.$$

This equation is the three-dimensional form of the equation satisfied by the stress function introduced by S. G. Lehnickiĭ [Z. Prikl. Mat. Meh. (N.S.) **4** (1940), no. 3, 43–60] for the solution of the symmetrical deformation of a body of revolution. The general solution of (1) may be obtained by expressing (u, v, w) in terms of two functions F and φ:

$$u=-\partial^2F/\partial x\partial z-\partial\varphi/\partial y,\quad v=-\partial^2F/\partial y\partial z+\partial\varphi/\partial x,$$
$$w=(B_{11}B_{33}^{-1})[\partial^2F/\partial x^2+\partial^2F/\partial y^2+(B_{44}B_{11}^{-1})\partial^2F/\partial z^2],$$

where F satisfies (2), and φ satisfies the equation

$$\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2+(B_{44}B_{66}^{-1})\partial^2\varphi/\partial z^2=0.$$

In cylindrical coordinate system (r, ϑ, z) the general solution may be obtained by a vectorial transformation.

From the English summary (Zbl **52**, 205).

Callahan, Willie Russell. On the flexural vibrations of circular and elliptical plates. Quart. Appl. Math. **13** (1956), 371–380.

R. D. Mindlin's equations for the bending of flat plates [J. Appl. Mech. **18** (1951), 31–38], which do not neglect transverse shear and rotary inertia, are expressed in general orthogonal curvilinear coordinates. In polar coordinates product solutions are obtained which can be made to satisfy various edge conditions. Eight different sets of boundary conditions are treated and the corresponding frequency equations given. In elliptic coordinates the variables can again be separated but simple product solutions are unable to satisfy certain edge conditions, notably those of a free edge. *R. C. T. Smith.*

★**Nishimura, Toru. Studies on vibration problems of flat plates by means of difference calculus. (Vibration of a square plate supported at four corners and being free along all edges.)** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 417–420. Science Council of Japan, Tokyo, 1954.

Livesley, R. K. The equivalence of continuous and discrete mass distributions in certain vibration problems. Quart. J. Mech. Appl. Math. **8** (1955), 353–360.

Several numerical methods for determining the natural frequencies of non-uniform shafts in torsional and transverse vibration require the replacement of any continuous distribution of matter by an "equivalent system", in which the mass of inertia is concentrated at particular points. This paper discusses the error introduced by this process by considering a few simple cases, in which both the continuous and the discrete systems can be treated exactly. Following a brief statement of the results obtained by previous authors, an expression is derived for the frequency error in the case of a uniform simply supported beam in transverse vibration. It is shown that if the continuous mass distribution of the beam is replaced by a set of equally spaced concentrated masses, the errors in the natural frequencies are proportional to the inverse fourth power of the number of segments into which the beam is divided. *Author's summary.*

Mettler, E. Nichtlineare Schwingungen und kinetische Instabilität bei Saiten und Stäben. Ing.-Arch. **23** (1955) 354–364.

The author modifies the usual linear theory of transverse vibrations of an elastic beam by including the nonlinear effect on the longitudinal tension due to the lengthening of the beam in the course of the vibration. He shows that when the tension is subject to a periodic fluctuation, the lateral displacement satisfies an equation like Mathieu's but with an extra nonlinear term. He then discusses the regions of stability of this equation in the space of the equation parameters, both in the linear case when the effect of the lengthening on the tension is neglected, and in the nonlinear case. Next, vibrations of an I-beam are studied, both bending and rotation of the beam being considered. The ends of the beam are fixed. The nonlinear lengthening term is also included. A coupled system of two equations of nonlinear Mathieu type is obtained. The stability of this system is discussed in the linear and nonlinear cases. *E. Pinney.*

Stoneley, R. The propagation of surface elastic waves in a cubic crystal. Proc. Roy. Soc. London. Ser. A. **232** (1955), 447–458.

The paper discusses the possibility of the propagation of elastic waves, analogous to Rayleigh waves or Love waves, over a (0, 0, 1) surface of a cubic crystal. An examination of symmetrical cases in which the direction of propagation is parallel to the x-axis or makes an angle of 45° with it shows that waves with amplitude falling off exponentially with distance from the free face do not exist for every set of values of the three elastic constants chosen at random. For crystals of aluminium and copper these Rayleigh-type waves do not exist, but for rock salt their existence is demonstrated both for the symmetrical cases and for an asymmetrical case. In the latter the particles describe ellipses in a plane inclined to the direction of propagation. Waves of Love type can exist only in the two symmetrical cases cited; it is shown that then the general Rayleigh-type motion degenerates into a superposition of Rayleigh waves and Love waves. It is easily seen that for all types of crystals the general result will be the same, but that the algebraical and arithmetical work involved will be extremely heavy. (Author's summary.) *E. Pinney* (Berkeley, Calif.).

Alekseev, A. S. Some laws of propagation of waves in a nonhomogeneous medium. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 989–992. (Russian)

The author studies the propagation of torsional waves in a semi-infinite circular cylinder of infinite radius, the rigidity of which is proportional to $(1+\varepsilon z)^2$, where ε is constant and (r, θ, z) are cylindrical coordinates. On the plane end $z=0$ the conditions $u=0$ and $\partial u/\partial z=F(r, t)$, where u is the displacement component in the direction of θ, are to be satisfied. Conditions which $F(r, t)$ must satisfy in order that the solution can be written as a certain integral with a Bessel kernel are given, and a particular case is discussed from the viewpoint of its physical interpretation. *R. N. Goss* (San Diego, Calif.).

Jobert, N. Effet de la courbure de la terre sur les ondes de Love. Ann. Géophys. **11** (1955), 1–48.

The earth is considered to be a homogeneous elastic spherical core surmounted by a homogeneous shell or mantle with different elastic parameters. Shear waves alone are considered. The boundary conditions are that

the surface tractions must be zero on the mantle surface and continuous across the core-mantle interface. A source inside the mantle is considered. The wave equation is separable in spherical coordinates, so a solution is obtained without difficulty in the harmonic case. Then the theory for a source of step-function type is obtained by a Fourier synthesis.

Watson's method is used to transform the series solution to a better form by first transforming it to a contour integral. A number of features of the resulting expression are studied, each requiring a more-or-less involved reduction. The motion of the surface of the body is traced in detail and is compared with that in the plane-surface case corresponding to a sphere of infinite radius.

E. Pinney (Berkeley, Calif.).

★ **Kakuzen, Mutsuo. A solution of fundamental equation of plasticity in polar coordinates.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 51–54. Science Council of Japan, Tokyo, 1954.

The equations of equilibrium and the yield condition for plane plastic flow are discussed in polar coordinates. No specific boundary-value problems are solved.

W. Prager (Providence, R.I.).

Prager, William. The sign of plastic power in the graphical treatment of problems of plane plastic flow. Quart. Appl. Math. **13** (1955), 333–335.

The use of hodographs in the analysis of an ideal plastic-rigid body undergoing plane strain has been described by A. P. Green [Philos. Mag. (7) **42** (1951), 900–918; Quart. J. Mech. Appl. Math. **6** (1953), 223–239; J. Mech. Phys. Solids **2** (1953), 73–80; MR **13**, 303; **14**, 1041; **16**, 94] and by W. Prager [Trans. Roy. Inst. Tech. Stockholm no. **65** (1953); MR **15**, 77]. The solution of a problem found in this way must necessarily be regarded as tentative until it has been checked that the rate of plastic work, $\dot{\gamma}$ (k=yield stress in shear, $\dot{\gamma}$=maximum engineering shear strain-rate), is everywhere non-negative, and then, and only then, may the solution be accepted as correct. A. P. Green (loc. cit.) has given one criterion for the sign of $\dot{\gamma}$ which involves the radii of curvature of the slip-line and hodograph fields. In the present paper, Prager gives an alternative criterion that involves the velocity components. (The final formula (4) involves an obvious misprint.) *H. G. Hopkins* (Sevenoaks).

★ **Yamamoto, Yoshiyuki. A general theory on the plastic behavior of metals.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 93–96. Science Council of Japan, Tokyo, 1952.

The mathematical structure of (incremental) stress-strain relations for a work-hardening plastic solid is investigated under the general assumption that the plastic strain increment is determined by the instantaneous stress and temperature, the plastic history, and the increments of stress and temperature. The usual geometric discussion in a nine-dimensional stress-space is generalized by providing a tenth dimension for the temperature, and the usual special assumptions regarding isothermal changes are supplemented by the following assumptions concerning temperature effects: 1) the material does not deform plastically at the absolute temperature zero; 2) a temperature change occurring under constant mean normal stress does not produce any plastic deformation; and 3) the effect of temperature increment on the plastic strain is equivalent to the effect of an appropriate additional stress increment. *W. Prager* (Providence, R.I.).

Šapiro, G. S. Propagation of elastic-plastic waves in bars of variable cross-section. Prikl. Mat. Meh. **16** (1952), 335–340. (Russian)

A general discussion, based upon characteristics theory, is given of the propagation of longitudinal elastic-plastic waves in thin bars of variable cross-sectional area. The stress-strain curve is taken to be approximated by two straight-line segments. The results are specialised to the case of a bar with linear taper. The paper also contains a brief discussion of the propagation of plane elastic-plastic waves in an infinite region. *H. G. Hopkins.*

Kristesku [Cristescu], N. Some remarks concerning the propagation of plastic waves in plates (axially symmetric case). Prikl. Mat. Meh. **19** (1955), 433–442. (Russian)

This paper concerns the velocity of propagation of plastic waves in thin uniform plates under conditions of rotational symmetry. Discussion is given of cases in which the material obeys either total or incremental stress-strain relations. *H. G. Hopkins* (Sevenoaks).

★ **Foulkes, J. Linear programming and structural design.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 177–184. National Bureau of Standards, Washington, D.C., 1955.

Using the principles of plastic design (or limit analysis), the author shows that designing a structural frame for minimum material consumption is equivalent to solving a linear programming problem. When the frame has to support only a single set of loads, this problem is of conventional type; when the frame is to be designed for a multiparameter family of loading conditions, the right-hand side of each subsidiary inequality of the linear-programming problem is itself obtained as the solution of an extremum problem. *W. Prager* (Providence, R.I.).

Chien, Wei-Zang, and Chen, Chih-Ta. Theory of rolling. Chinese J. Phys. **9** (1953), 57–92. (Chinese. English summary)

The theory of rolling given in the present paper is based upon the following assumptions: (1) There is no lateral spreading. (2) The material used is incompressible, isotropic, and viscous. It also obeys von Mises-Henky's condition of yielding. (3) The elastic deformation of rolls is neglected. (4) The contact region between the material and rolls is divided into three parts: namely the forward slipping region, sticking region, and backward slipping region. The conditions of continuity of velocity and pressure are used. The results obtained from this theory agree very well with the experimental results given by Siebel and Lueg in 1933. *Autoreferat* (Zbl **52**, 209).

★ **Inoue, Nobuo. Statically determinate solutions of elastic-plastic problem.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 245–250. Science Council of Japan, Tokyo, 1952.

Al'perin, L. B. Passive pressure of sand against a retaining wall. Inžen. Sb. **21** (1955), 156–162. (Russian)

The author follows closely the approach used by v. Karman for the case of active pressure [Verh. 2. Internat. Kongresses Tech. Mech., Zürich, 1926, Füssli, Zürich,

1927, pp. 23–32]. A statically admissible stress field is obtained by assuming the stress is proportional to the distance from the top of the wall. A numerical comparison is made with results from Coulomb's theory. The reviewer considers that the author's solution might be proved to be a lower bound by a proper extension; the Coulomb theory gives an upper bound. The results quoted are particularly interesting from this viewpoint because the two pressures approach each other for large negative surcharge angles. Since if the new solution is a lower bound the correct pressure must then lie in between and Coulomb's theory would be established as a good approximation over part of the range.

R. M. Haythornthwaite (Providence, R.I.).

See also: Abramowitz and Cahill, p. 413; Burgers, p. 418.

MATHEMATICAL PHYSICS

★**Pi Calleja, Pedro. The functional equations of the theory of magnitudes.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 199–280. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

A scholarly and detailed review of the theory of magnitudes as applied to dimensional analysis, developed following a new set of postulates and basic definitions. The conclusions cannot be summarized briefly; the orientation is philosophical. *G. Birkhoff* (Cambridge, Mass.).

Kron, Gabriel. Tearing and interconnecting as a form of transformation. Quart. Appl. Math. **13** (1955), 147–159.

A systematic procedure for the analysis of physical systems is proposed based upon "tearing" and "interconnecting." The term "tearing" refers to the introduction of variables of constraint which permit analysis of a complex system in simpler stages. The "interconnecting" concerns the synthesis of the partial solutions to obtain the solution of the original system. *C. Saltzer.*

See also: Page, p. 375; Arcidiacono, p. 437.

Optics, Electromagnetic Theory, Circuits

★**Bateman, H. The mathematical analysis of electrical and optical wave-motion on the basis of Maxwell's equations.** Dover Publications, Inc., New York, 1955. vii+159 pp. $ 1.60.

Photo-offset reprint of the original edition [Cambridge, 1915].

Vasil'kovskiĭ, A. A. On the theory of the optical computation of multilayered periodic plates. Ž. Tehn. Fiz. **25** (1955), 1326–1331. (Russian)

The amplitude coefficients for reflection and transmission of light on a system of homogeneous layers have the character of recurrence formulae. The author uses this property in the deduction of amplitude coefficients for a plate consisting of an arbitrary number of periodically repeated equal elements, formed by a system of a finite number of layers. By a suitable choice of two variables and by means of the theory of continued fractions the author gets relations enabling him to calculate the amplitude coefficients of the whole plate. *M. Brdička.*

Karp, S. N., and Russek, A. Diffraction by a wide slit. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-75** (1955), i+40 pp.

The two-dimensional problem of diffraction of electromagnetic waves by an infinite conducting plane with a slit is investigated for both plane-wave excitation and line-source excitation, the electric vector of the incident field being parallel to the edges of the slit. Approximate expressions for the near and far fields are derived in terms of the classical solutions for the fields obtaining if one of the two half-plane screens is removed, so that the author's procedure is in a sense closely related to the method followed by Schwarzschild [Math. Ann. **55** (1902), 177–247]. The diffracted field is expressed as the sum of a "non-interaction" term (identical with Schwarzschild's leading term) and an "interaction" term, conveniently expressed in terms of the response of a half-plane to a line source of appropriate amplitude placed at the opposite edge. Whereas Schwarzschild's series solution is mathematically convergent for all ratios: slit-width to wavelength, the result derived here is at best an asymptotic approximation expected to be good for large values of this ratio, and in fact better than Schwarzschild's leading term. As borne out by numerical calculations, the authors' procedure permits a rapid calculation of diffraction pattern and transmission cross-section for slits wider than one wavelength, the accuracy and rapidity increasing with slit width. For purposes of comparison, the authors calculated in a number of cases the corresponding mathematically exact values obtained from the Mathieu-function series solution. In two tables numerical results are presented for the case of plane-wave excitation at normal incidence. *C. J. Bouwkamp* (Eindhoven).

Šantavý, Ivan. The radiation of an electric dipole near the boundary between two media. Czechoslovak J. Phys. **5** (1955), 340–368. (Russian. English summary)

This paper is another contribution to the literature on a famous problem of Sommerfeld. The distant field of a radiating vertical dipole near the horizontal plane boundary separating two non-conducting semi-infinite media is studied. The dielectric constants are assumed to be different, but the permeability is the same in the two media. The distinctive feature here is that the time dependence of the dipole moment is not restricted to be sinusoidal; otherwise the problem is formulated in the usual way. For the solution the author uses a method attributed to L. Cagniard [Réflexion et réfraction des ondes séismiques progressives, Gauthier-Villars, Paris, 1939], in which the separation of the time variable t in the plane-wave equation satisfied by the Hertz potential is made in the form $e^{st}X$. Here s is a positive parameter, and X is a function of the cylindrical coordinates and s having three components analogous to Sommerfeld's Q_1, Q_2, and P. The potential is then given explicitly in terms of the inverse Laplace transform of X, found by putting into the form of a Laplace transform the integral representation of X in terms of plane waves. The form of the wave fronts is determined for the two cases where the dielectric constant in the medium containing the dipole

is respectively greater than and less than the constant in the other medium. The author shows that his solution contains the harmonic time-dependent solution as a special case. *R. N. Goss* (San Diego, Calif.).

Rubinowicz, A. Fortpflanzung von Sprüngen elektromagnetischer Feldstärken und Eindeutigkeitsbeweis für das Anfangswertproblem der Maxwellschen Gleichungen. Acta Phys. Polon. **14** (1955), 209–224. (Russian summary)

The discontinuities considered in this paper are jumps in the values of the components of the electric and magnetic vectors across a surface which is moving with an arbitrary velocity in its normal direction. An equation for the balance of energy is obtained; it follows that such a surface of discontinuity must carry an electric and magnetic current sheet. But if these currents vanish, then the moving surface, regarded as a hypersurface in a space-time manifold, must be a characteristic hypersurface of Maxwell's equations.

The uniqueness of the solution of Maxwell's equations which satisfies given intial conditions and which admits discontinuities of the type discussed in the first part of the paper, is proved.

It is not possible to give more details of the work in a brief space. *E. T. Copson* (St. Andrews).

Rubinowicz, A. Der Satz von der Erhaltung des Impulses und die Fortpflanzung von Sprüngen elektromagnetischer Feldstärken. Acta Phys. Polon, **14** (1955), 225–231. (Russian summary)

Two forms for the momentum-energy tensor for an electromagnetic field satisfying Maxwell's equations

$$\varepsilon\frac{\partial \mathbf{E}}{d\tau}=\operatorname{curl}\mathbf{H},\ \mu\frac{\partial \mathbf{H}}{\partial\tau}=-\operatorname{curl}\mathbf{E},\ \operatorname{div}(\varepsilon\mathbf{E})=0,\ \operatorname{div}(\mu\mathbf{H})=0$$

where $\tau=ct$, have been proposed. Both are given by the scheme

$-T$	$\frac{ip}{c}\mathbf{S}$
$\frac{ip}{c}\mathbf{S}$	$-W$

where T is the Maxwell stress tensor, W the energy density, $\mathbf{S}$ Poynting's vector; in Minkowski's form, $p=\varepsilon\mu$, in Abraham's, $p=1$.

It is shown here that the assumption, that a discontinuity in the electric and magnetic vectors across a surface moving with the velocity of light $c/\sqrt{(\varepsilon\mu)}$ makes no contribution to the momentum-balance equation in space-time, is compatible with Minkowski's form but not with Abraham's. *E. T. Copson* (St. Andrews).

Poincelot, Paul. Réflexion d'une onde électromagnétique plane sur un gaz ionisé et stratifié. C. R. Acad. Sci. Paris **241** (1955), 1272–1275.

The refractive index n of the author's previous work [same C.R. **241** (1955), 186–188, 290–292, 649–651; MR **17**, 216] is generalized to the form $n=1$ $(x<0)$, $n=[1-(a^2/\omega^2)(1-e^{-ax})]^{\frac{1}{2}}$ $(x>0)$. *R. N. Goss.*

Budden, K. G. The numerical solution of the differential equations governing the reflexion of long radio waves from the ionosphere. II. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1955), 45–72.

"This paper presents a series of curves which are the results of some calculations of the reflecting properties of various models of the ionosphere for radio waves of frequency 16 kc/s. The method of calculation was described in a previous paper [Proc. Roy. Soc. London. Ser. A. **227** (1955), 516–537; MR **16**, 752]. No attempt is made to deduce a model of the ionosphere capable of explaining all the observations, but the aim has been rather to establish some general principles which may indicate how future theoretical and experimental work should be planned. In most of the calculations it was assumed that the earth's magnetic field is vertical and that the electron collision frequency in the ionosphere is constant. The limitations imposed by these restrictions are discussed.

The first half of the paper describes some calculations for a model of the ionosphere in which the electron density increases exponentially with height, and the second half deals with a model having both *D*- and *E*-layers. The results in both cases are compared with observations." *E. Isaacson* (New York, N.Y.).

Teixeira, José Gaspar. The dynamical law of free electricity. Gaz. Mat., Lisboa **15** (1955), no. 60–61, 17-22. (Portuguese)

Leccas, Daniel M. Determination of mechanical characteristics of electrical machinery. Prakt. Akad. Athēnōn **29** (1954), 16–25 (1955). (Greek. French summary)

This paper deals with the mathematical analysis of electrical machinary based on Kron's theory. In the first part the author derives relations between voltages and currents involving the characteristic quantities of the system, i.e., the self and mutual inductances and capacitances and the mechanical quantities (angular velocity, etc.) associated with electrical machinary, in tensorial form. The results are then applied to the following systems: (a) single-phase systems in series; (b) synchronous motors; (c) asychronous polyphase systems and finally shunt motors. *N. Chako* (New York, N.Y.).

Neĭmark, Yu. I. On auto-oscillations and forced oscillations of relay systems with retardation. Avtomat. i Telemeh. **16** (1955), 225–232. (Russian)

The author utilizes discrete idealizations and considers the transfer coefficient $K(p)=K_0(p)+K_1(p)e^{-\alpha_1 p}+\cdots+K_s(p)e^{-\alpha_s p}$, where $K_i(p)$ are quotients of polynomials and α_i are time delays. Several cases are considered and some examples given. *N. Levinson* (Cambridge, Mass.).

Bogatyrev, O. M. A general method of solution of problems of linear networks with variable resistances in the branches. Élektričestvo **1955**, no. 9, 67–69. (Russian)

Networks are considered containing one or more variable resistances in its branches, the networks being otherwise linear. The superposition theorem can be applied to the linear part of the network. The linear part of the network containing internal generators can be represented by its Thevenin equivalent. The variation is derived of the terminal voltages and currents caused by one or two varying resistances. The generalized Thevenin theorem for networks with varying resistances derived in a previous article [Meerovič, Élektričestvo **1951**, no. 9, 67] is shown to be a consequence of the Thevenin principle applied to the linear part of the network. *H. A. Haus.*

Vasilache, Sergiu. Nouvelle méthode de calcul des réseaux électriques à paramètres variables. An. Acad. R. P. Române. Secţ, Şti. Mat. Fiz. Chim. Ser. A. 3 (1950), 305–324. (Romanian. Russian and French summaries)

Verf. stellt sich die Aufgabe, die Gleichungen

$$\sum_{i=1}^{n}\left\{\frac{d}{dt}(L_{ki}i_i)+R_{ki}i_i+\frac{1}{C_{ki}}\int_0^t i_i(s)ds\right\}=E_k(t)\quad(k=1,\cdots,n)$$

für die Ströme $i_i=i_i(t)$ in den n unabhängigen Stromkreisen eines Netzwerks bei den Anfangsbedingungen $i_i(0)=0$ für zeitabhängige Schaltelemente $L_{ki}(t)$, $R_{ki}(t)$, $C_{ki}(t)$ zu lösen. Er bringt sie dazu mit den Abkürzungen

$$u_i=u_i(t)=\frac{di_i}{dt},\quad K_{ki}(t,s)=\frac{dL_{ki}(t)}{dt}+R_{ki}(t)+\frac{1}{C_{ki}(t)}(t-s)$$

in die Gestalt

$$(*)\quad \sum_{i=1}^{n}L_{ki}u_i(t)+\int_0^t\sum_{i=1}^{n}K_{ki}(t,s)u_i(s)ds=E_k(t)\quad(k=1,\cdots,n),$$

löst unter der Annahme einer nichtverschwindenden Determinante $|L_{ki}|$ nach den $u_i(t)$ des ersten Summanden auf und gibt die Lösung dieses Volterraschen Integralgleichungssystems in bekannter Weise in der Gestalt

$$u_k(s)=f_k(s)+\int_0^t\sum_{i=1}^{n}S_{ki}(s_1,s)f_i(s_1)ds_1,\quad i_k(t)=\int_0^t u_k(s)ds,$$

wo die Funktionen $f_k(s)$, $S_{ki}(s_1,s)$ aus den in (*) auftretenden bekannten Funktionen gewonnen sind; der lösende Kern $S_{ki}(s_1,s)$ ist dabei wie üblich als eine in einem passenden Gebiet konvergente unendliche Reihe gebildet. Auf das Auftreten ähnlich gebauter Gleichungssysteme bei der Behandlung von Nachwirkungsproblemen wird hingewiesen. *A. Stöhr* (Zbl 41, 553).

Vasilache, Sergiu. Une nouvelle méthode de calcul des réseaux électriques à paramètres variables. II. Applications. An. Acad. R. P. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 3 (1950), 507–530. (Romanian. Russian and French summaries)

In der vorstehend referierten Arbeit hatte Verf. die Behandlung von Netzwerken mit zeitabhängigen Parametern auf Integrodifferentialgleichungen zurückgeführt. In der vorliegenden Arbeit bespricht er einige Lösungsmethoden dafür. Besonders wird der Fall eines einzigen Stromkreises mit L, R, C und einer Wechselspannungsquelle in Serie betrachtet, wobei L $L(t)$ $a+b\cos\Omega t$ ist, R und C dagegen zeitunabhängig sind; als Modell dafür kann ein Stromkreis angesehen werden, der einen mit vorgegebener Drehzahl laufenden Motor enthält. Die Gleichungen dafür führen nach einigen Transformationen und Vernachlässigungen auf eine Differentialgleichung vom Mathieuschen Typus. Die unter Benutzung von Laplacetransformationen vorgenommene Auflösung läßt das Auftreten von Resonanzphänomenen sowie die Aufteilung in einen periodischen Vorgang und einen Einschwingvorgang erkennen. *A. Stöhr* (Zbl 41, 553).

Yeh, Kai-Yuan. Electric circuit analysis by the method of difference equation. Chinese J. Phys. 9 (1953), 192–200. (Chinese. English summary)

The method of difference equation recently employed by Chien Wei-Zang [same vol., 170–182; MR 17, 430] in his treatment of continuous beams is applied here to the solution of electric circuit problem. The present paper is confined to the problem of the continuous network (I) in steady state, (II) in transient state. In (I), the chief result is

$$v_x=P_0\alpha_x+Q_0\beta_x+\sum_{i=1}^{x}\frac{F_{i+1}}{q_{i+1}}\frac{\beta_i\alpha_x-\alpha_i\beta_x}{\alpha_{i-1}\beta_i-\alpha_i\beta_{i-1}},$$

where v_x is the voltage at the point x, α_x and β_x are defined by $\alpha_{x+2}=p_{x+2}\alpha_{x+1}+q_{x+2}\alpha_x$, $\beta_{x+2}=p_{x+2}\beta_{x+1}+q_{x+2}\beta_x$, $\alpha_0=\beta_1=0$, $\alpha_1=\beta_0=1$; p_x, q_x, F_x are defined by

$$p_x=1+\frac{Z_{x-1}}{Z_{x-1}'}+\frac{Z_{x-1}}{Z_{x-2}},\quad q_x=-\frac{Z_{x-1}}{Z_{x-2}},$$

$$F_x=e_{x-1}-e_{x-2}\frac{Z_{x-1}}{Z_{x-2}}-e_{x-1}'\frac{Z_{x-1}}{Z_{x-1}'};$$

e_x, Z_x and e_x', Z_x' are the applied voltage and impedance between the nodes x, $x+1$ and x, x' respectively. P_0 and Q_0 are integration constants, which can be determined by the terminal conditions. In (II), we use the Laplace transformation to solve the problem of the electric circuit in transient state. At first, we transform the circuit into one in an equivalent steady state and hence obtain the equivalent steady voltage with the previous method. Then we get the required solution by means of the inverse Laplace transformation. *Autoreferat* (Zbl 52, 217).

Fialkow, Aaron, and Gerst, Irving. Impedance synthesis without minimization. J. Math. Phys. 34 (1955), 160–168.

In this paper the authors present a new impedance synthesis procedure without ideal transformers. This procedure is based on the authors' theorem [which is a generalization of a theorem by P. I. Richards, Duke Math. J. 14 (1947), 777–786; MR 9, 181] that if p_1, p_2, $\cdots$, p_s are complex numbers with positive real part such that non-real p_i occur in conjugate pairs and if Z is a positive real function of degree n and $Z(p_i)=c$ $(i=1,2,\cdots,s)$, c real and positive, then the function Z' given by

$$Z'(p)=\frac{Z_1(p)Z(p)-c}{cZ_1(p)-Z(p)}$$

is a positive real function of degree not exceeding n where

$$Z_1(p)=\frac{\prod_{i=1}^{s}(p+p_i)+\prod_{i=1}^{s}(p-p_i)}{\prod_{i=1}^{s}(p+p_i)-\prod_{i=1}^{s}(p-p_i)}$$

is a reactance. From this theorem, the authors show that the degree of Z' is $n-u$ if and only if u of the p_i are roots of the equation $Z(p)+Z(-p)=0$. Since from the theorem $Z(p)=c(Z_1(p)Z'(p)+1)(Z_1(p)+Z'(p))^{-1}$ which is realized by two networks in parallel where one network consists of cZ_1 and c/Z' in series and the other cZ' and c/Z_1 in series, the synthesis procedure is completely defined.

In the rest of the paper the authors consider details of the reduction procedure and ways by which computation is simplified; e.g., the authors show that the equation $Z(p)+Z(-p)=0$ need be solved only for the initial Z. The authors note that at present the increased simplicity of computation of their method is achieved at the expense of an increase in the number of network elements in the general case. *C. Y. Lee* (Chatham, N.J.).

Sirlin, A., and Amati, D. Contribution to the theory of ferroresonance. Rev. Un. Mat. Argentina 16 (1955), 91–123. (Spanish. English summary)

Forced oscillations in series and parallel circuits containing resistances, capacitances and an iron-core in-

ductance are discussed. The authors use limited Fourier developments for the unknown; it is claimed that the results are accurate even for relatively strong non-linearities. The non-existence of subharmonic vibrations and the stability properties are also studied. The discussion confirms the classical typical behavior of these circuits (hysteresis). *J. L. Massera* (Montevideo).

★**Higonnet, R., et Gréa, R. Etude logique des circuits électriques et des systèmes binaires.** Editions Berger-Levrault, Paris, 1955. viii+452 pp.

This up-to-date profusely illustrated book is devoted to the application of boolean algebra to switching circuits. The table of contents includes: I. Study of binary combinations. II. Relays and commutators. III. Application of logical algebra to 2-pole switching circuits. IV. Tables for reduction of algebraic expressions. V. Representation of combinations by geometric figures and tables. VI. Reduction of circuits beyond their algebraic and geometric representation. VII. Study of permutations and symmetries. VIII. Associating circuits to circuit types. IX. Circuits closed when M out of N relays are active. X. Multipole circuits. XI. Control by codes. XII. Control by switching circuits. XIII. Sequential or temporal circuits, control of a succession of operations. XIV. Circuits for step-by-step rotating commutators. XV. Function time for relays. XVI. Some applications of relay circuits. XVII. Application of logical algebra to rectifiers and vacuum tubes. XVIII. Application of logical algebra to combinations of mechanisms having two positions (railroad switching and signaling). XIX. Application to typographical composition. XX. Catalogue of circuit types for 3 and 4 relays. A. French-English Vocabulary (very helpful 8 pp. since it treats specialized electrical terms not available in ordinary dictionaries).

S. Sherman (Philadelphia, Pa.).

Mealy, George H. A method for synthesizing sequential circuits. Bell System Tech. J. **34** (1955), 1045–1079.

This paper describes a method for designing switching circuits using relays or other logical switching elements. An abstract model of the circuit is defined which has inputs, outputs and internal states. Changes of the signals at points in the circuit are assumed to take place sequentially, each set of changes following the previous set after a finite time interval. The behaviour of the circuit is completely described by specifying the outputs and the internal state after the next time interval as functions of the inputs and the present internal state. This specification is given in three ways—as a set of Boolean functions, as a state diagram, and as a truth table. The state diagram is a graphical way of representing these relationships in which nodes represent internal states and lines represent transitions caused by given input combinations. A truth table contains this information in tabular form. Unpublished results of E. F. Moore are quoted in which equivalence between two abstract models is defined and a method given for finding a model equivalent to any given model and having a minimum number of internal states. Moore's method for reducing the number of states is generalized to apply to circuits which may have redundant or unused states at the expense of the minimality property. The author advocates using the generalization of Moore's method combined with a method due to Huffman [J. Franklin Inst. **257** (1954), 161–190, 275–303; MR **15**, 1009] which is simpler to use but which may achieve less reduction of the number of internal states if used alone.

A method is given for obtaining an initial-state diagram or truth table provided an internal state exists to which the circuit always returns after a fixed number of time intervals. This diagram may then be reduced by the method mentioned above. Final realization of the circuit in terms of logical gates or relays may then be carried out by use of standard techniques which are reviewed in the paper. Numerous examples are given and some attention is paid to non-clocked, or asynchronous relay circuitry.

D. E. Muller (Urbana, Ill.).

Craven, T. L. An engineering application of Boolean algebra. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 121–130.

Taft, V. A. Necessary and sufficient conditions for the physical realization of a matrix of impedances (admittances) of order $2m$ in the form of a passive multipole. Dokl. Akad. Nauk SSSR. (N.S.) **84** (1952), 499–501. (Russian)

The necessary condition for the physical realizability of a matrix of impedances or admittances of order $2m$ as a passive $2m$-pole is that the matrix be a positive real matrix function. A proof that this condition is also sufficient is given by showing how the desired network is synthesized.

C. Saltzer (Cleveland, Ohio).

Hohn, Franz E., and Schissler, L. Robert. Boolean matrices and the design of combinational relay switching circuits. Bell System Tech. J. **34** (1955), 177–202.

If the Boolean function which gives the admittance between the ith and jth nodes of a contact network is regarded as the element in the ith row and jth column of a matrix, a generalization of the notion of a switching function is obtained. Some results of Lunts [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 421–423; MR **11**, 574] concerning these switching matrices are restated and proved. The fundamental theorem dealing with the relation between a connection matrix and its reduction is generalized and a necessary and sufficient condition is given that a switching matrix be an output matrix. Techniques based on matrix operations are given for analysis and synthesis of contact networks.

C. Saltzer (Cleveland, Ohio).

Samson, Edward W., and Mills, Burton E. Circuit minimization: algebra and algorithms for new Boolean canonical expressions. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., Tech. Rep. **54-21**, 54 pp. (1954).

An integral-valued function of Boolean expressions is suggested as a measure of the complexity of circuits of bistable elements. Canonical forms and operators are defined, and an operator algebra is developed for studying the minimization of the above function, the Boolean length. Algorithms based on these operators are presented with examples. *C. Saltzer.*

Povarov, G. N. On the synthesis of contact multipoles. Dokl. Akad. Nauk SSSR (N.S.) **94** (1954), 1075–1078; erratum **96** (1954), 1084. (Russian)

A (p, q)-pole contact network has p inputs and q outputs. If the outputs of a (p, q)-pole are connected to the inputs of an (r, s)-pole, the resulting network is a (p, s)-pole. The methods of the paper are based on a generalization of Shannon's theorem for $(1, q)$-pole net-

works [Bell System Tech. J. **28** (1949), 59–98; MR **10**, 671] and give the Boolean admittances of the composite (p, s)-pole network in terms of the admittances of the constituent networks by a rule like that of matrix multiplication. Synthesis procedures based on this theorem are given and corresponding bounds for the number of contacts required are computed. *C. Saltzer.*

See also: Tasny-Tschiassny, p. 412.

Quantum Mechanics

★**Landé, Alfred. Foundations of quantum theory. A study in continuity and symmetry.** Yale University Press, New Haven, 1955. viii+106 pp. $ 4.00.

This small book has as its aim to develop the foundations of quantum theory not by reference to specific experiments on microscopic systems but rather as consequences of general physical principles of continuity and symmetry. In Ch. I, "Entropy, continuity and quantum structure", various concepts and rules of quantum theory are developed from the requirement of entropy continuity. In Ch. II, "Superposition mechanics and symmetry principles", the mechanics of superposition, with its addition of probability amplitudes, is developed on the basis of the further requirement that there exist a general correlation law between various states, symmetrical in all states. The symmetry properties of the probability amplitudes, or wave functions, are also determined by means of the continuity principle. Ch. III, "Quantum dynamics", deals with conjugate dynamical variables on the basis of a new definition of conjugacy and discusses various topics of quantum dynamics, such as wave mechanics. As the author points out in the preface, the book is not an introduction for beginners but rather a postscript to more traditional books on quantum mechanics. *N. Rosen* (Haifa).

★**Döring, Werner. Einführung in die Quantenmechanik.** Vandenhoeck & Ruprecht, Göttingen, 1955. 517 pp. DM 26.00.

This book presents an introduction to non-relativistic quantum mechanics. It consists of four parts. In Part I, The wave mechanics of stationary states, the reader is introduced to the time-independent Schrödinger equation and then to the Hamiltonian operator. The Schrödinger equation is solved in the case of the harmonic oscillator and the hydrogen atom. Expansion in eigenfunctions, perturbation theory and the matrix formulation of quantum mechanics are also discussed. Part II, The wave mechanics of non-stationary states, deals mainly with the time-dependent Schrödinger equation and its application to transitions and to the dispersion and absorption of light. Part III, General quantum mechanics of a single particle, treats a variety of topics including probability relations, the uncertainty principle, the abstract formulation of quantum mechanics and the solution of problems without the use of a particular representation, spin and angular momentum, and time-dependent operators. Part IV, The quantum mechanics of a many-particle system, considers both the quantization of the particle picture, which gives the Schrödinger equation in configuration space, and the quantization of the wave picture, or second quantization. It includes discussions of the helium atom, multiplet spectra, the hydrogen molecule and scattering.

The author has restricted himself to selected topics, which are treated carefully and in considerable detail. It is assumed that the reader is familiar with classical physics and the Bohr theory and has a mathematics background that includes differential and integral calculus and a knowledge of the basic results of the theory of differential equations and of function theory. The special functions of mathematical physics are not assumed to be known, and their properties are developed as they are needed. *N. Rosen* (Haifa).

Costa de Beauregard, O. Covariance relativiste à la base de la mécanique quantique. J. Phys. Radium (8) **16** (1955), 770–780.

Presents a convenient notation for expressing and deriving, in a covariant manner, several well-known results involving Fourier transforms of the wave functions of relativistic free particles. The transforms in question are essentially three-dimensional and hence are usually discussed in noncovariant language. Use is made of arbitrary space-like surfaces, and included are such topics as the covariant formulation of Parseval's theorem, occupation numbers, the separation of positive and negative frequencies and Schwinger's formal solution of the Cauchy initial-value problem [Phys. Rev. (2) **74** (1948), 1439–1461, eqs. (2.22, 2.23); MR **10**, 345]. The presentation applies to particles of arbitrary spin. The paper ends with several long paragraphs discussing familiar difficulties, both conceptual and mathematical, which arise when external fields are applied, leading to four-dimensional Fourier integrals. Some elementary remarks are made concerning the Heisenberg and interaction representations and Feynman's formulation of quantum field theory. *B. S. DeWitt* (Chapel Hill, N.C.).

Arcidiacono, Giuseppe. Sul gruppo aggiunto del gruppo di Galileo. Rend. Mat. e Appl. (5) **14** (1955), 633–654.

Explicit and detailed features of the adjoint representation of the ten-parameter 'Galilean' group generated by translations in space-time, rotations in space, and transformation to uniformly accelerated axes, is investigated, in continuation of work of G. Leti (not yet published) and along general lines due to Fantappiè [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12** (1952), 285–290, 553–558; MR **14**, 117, 339]. *I. E. Segal.*

Biswas, S. N. Fredholm theory and Hankel transform of the Schrödinger equation. Indian J. Phys. **29** (1955), 309–318.

If $\phi(p)$ is the Hankel transform of the wave function, the Schrödinger equation for scattering in a Yukuwa potential can be written as

$$(p^2-k^2)\phi(p)+\lambda\int_0^\infty V(p, p')\phi(p')dp'=0.$$

Writing $\phi(p)=\delta(p-k)-(p^2-k^2)^{-1}B(p)$, the above gives a (Fredholm) integral equation for $B(p)$. The phase shifts are given directly by the formula $\tan\eta=(\pi/2k)B(k)$. The author makes a numerical approximation to the Fredholm solution for the S phases. *A. Salam.*

Laforgue, Alexandre. Extension du calcul d'erreur à l'ensemble des états d'un système stationnaire. C. R. Acad. Sci. Paris **240** (1955), 2122–2124.

As a measure of the approximation of two normalized wave functions f and g to each other, the author introduces ω defined by $2\cos\omega=\langle f|g\rangle+\langle g|f\rangle$. He here extends a previous paper dealing with the ground state [same C.R.

236 (1953), 1356–1358] to obtain limits for the error ω when excited states are approximated by the method of Ritz. *A. J. Coleman* (Toronto, Ont.).

Gürsey, Feza. One-dimensional motion of the electron in relativistic wave mechanics. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 161–165 (1955). (Turkish summary)

The subject of the paper is the three-dimensional Dirac equation in a field characterized by a scalar potential which is a function of only one variable. In this case the Dirac equation can be transformed into a one-dimensional orm depending only on the variable contained in the potential. This transformation is given explicitly. It is also shown that the solution of this reduced form of the Dirac equation can be derived from the complex solution of a first-order ordinary differential equation. The connection between the two solutions is given.

M. J. Moravcsik (Upton, N.Y.).

Corinaldesi, E. Construction of potentials from phase shift and binding energies of relativistic equations. Nuovo Cimento (9) **11** (1954), 468–478.

Given the S phase shift and binding energies, the potential $V(r)$ of the Klein-Gordon equation is determined following previous work by Jost and Kohn on the Schrödinger equation [Danske Vid. Selsk. Mat.-Fys. Medd. **27** (1953), no. 9; MR **15**, 125] and using the Gelfand-Levitan integral equations. *N. Levinson* (Cambridge, Mass.).

Morpurgo, G., Touschek, B. F., and Radicati, L. A. On time reversal. Nuovo Cimento (9) **12** (1954), 677–698.

In the extensive literature devoted to the problem of time-reversibility in quantum mechanics, this is one of the few papers which add substantially to the original discussion by E. P. Wigner [Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. **1932**, 546–559]. Section 1 deals with reversibility in classical mechanics. It is shown that the ordinary ideas of reversibility are embodied in the following definition. A system specified by a Hamiltonian H is reversible if H is invariant under a transformation

$$(1) \qquad r_i \to r_i,\ p_i \to -p_i + [\partial F/\partial r_i],$$

where F is some function of the position coordinates r_i.

In quantum mechanics, for a system specified by a time-independent Hermitian Hamiltonian H, there always exists a matrix K such that

$$(2) \qquad KH^*K^\dagger = H,$$

and the time-reversal of any Hermitian operator Q is then defined by

$$(3) \qquad \hat{Q} = KQ^*K^\dagger,$$

the time-reversal of a state ψ by

$$(4) \qquad \hat{\psi} = K\psi^*.$$

From these definitions it already follows that the scattering matrix S is reversible in the sense that

$$(5) \qquad \Phi^* S \Phi = \hat{\Phi}^* S \hat{\Phi}.$$

Thus the validity of (5), which is usually supposed to express some special property of time-reversible systems, is actually universal.

To express a non-vacuous notion of time-reversibility, the authors impose on K an additional condition, that the time-reversed operators agree with their classical counterparts in the sense of the correspondence principle. For a system of particles with positions r_i and spins σ_i, the extra condition on K is

$$(6) \qquad \hat{r}_i = r_i,\ \hat{\sigma}_i = -\sigma_i.$$

For a system of fields, the state of which can be described by occupation numbers N_k of particles with momentum k, the reversibility condition is

$$(7) \qquad \hat{N}_k = N_{-k}.$$

The conditions (6) and (7) are much less stringent than those customarily demanded of reversible systems. The authors argue cogently that the usual definitions of reversibility, which require (2) to hold with a special matrix K specified in advance, are not based on any physical necessity. Therefore all deductions, in which certain types of physical system have been excluded from consideration because they did not satisfy the old definitions of reversibility, need to be re-examined.

The authors state that in all interesting cases, where such arguments have been used to exclude certain possible interactions between particles, the exclusion can be maintained with their weaker definition of reversibility. Only the deduction is more difficult because all possible K need to be examined. As an example, they prove that the system of a Dirac field ψ coupled to a scalar field ϕ by a scalar and a vector interaction,

$$(8) \qquad H_{\mathrm{INT}} = g\psi^\dagger\beta\psi\phi + f[\psi^\dagger(\pi + \alpha\cdot\operatorname{grad}\phi)\psi],$$

is irreversible unless either $g=0$ or $f=0$. *F. J. Dyson.*

Morpurgo, G., and Touschek, B. F. Space and time reflection of observable and non-observable quantities in field theory. Nuovo Cimento (10) **1** (1955), 1159–1179.

This paper continues the argument of the paper reviewed above, the discussion being now restricted to reversibility in quantum field theory. The definition finally adopted for time-reversibility is that there should exist a K for which

$$(9) \qquad \hat{N}_q(Q) = N_q(\hat{Q}),$$

where $N_q(Q)$ denotes the occupation number for a particle-state specified by a set of eigenvalues q for a complete set of one-particle observables Q. The $\hat{Q}$ are the time-reversed Q, defined for the one-particle system by the correspondence principle, for example when Q is the momentum k we have $\hat{Q} = -k$. For a system to be time-reversible it is required that a K exist satisfying (2) of the preceding review and satisfying (9) for every choice of Q. The definition is thus stronger than that of the preceding review where (9) was required only for $Q=k$.

The following theorems are proved. (I) For any free field without interaction a special K_0 can be found satisfying (9). With this choice of K we have, for example, for a neutral scalar field φ

$$(10) \qquad \varphi(r) = -\varphi(r),$$

for a charged scalar field φ

$$(11) \qquad \varphi^*(r) = -\varphi(r),$$

and for a Dirac field ψ

$$(12) \qquad \psi^*(r) = i\alpha_1\alpha_2\alpha_3\beta\psi(r),$$

in agreement with older definitions of time-reversal. (II) For the neutral scalar field the most general K is

$$(13) \qquad K = \exp[iF(N)]K_0,$$

where N is the sum of all meson occupation numbers and F is an arbitrary Hermitian function. For the Dirac field the same result holds with $F(N)$ replaced by $F(N_+, N_-)$,

where $N_\pm$ are the sums of occupation numbers taken for positive and negative energy states separately. (III) For the system of nucleons, mesons and photons coupled by symmetric pseudoscalar and electromagnetic interactions, a special K_0 exists, identical with the K_0 defined for the same fields without interaction, and the most general K is given by

$$K = \exp[iF(N_\lambda, Q)]K_0, \tag{14}$$

where N_λ is the number of nucleons minus anti-nucleons, and Q is the total charge of the system. (IV) For any reversible system of interacting fields, all observable quantities (for example, quantities bilinear in ψ^* and ψ) transform in a determinate way under time-reversal, independent of the choice of K. But unobservable quantities (namely, operators like ψ which have matrix elements only between states separated from each other by an absolute selection rule) do not transform in a determinate way; their behavior under time-reversal can be changed by changing K.

It is a consequence of Theorem IV that the division of Fermi particles into types [C. N. Yang and J. Tiomno, Phys. Rev. (2) **79** (1950), 495–498; MR **12**, 227], by the transformation law of ψ under time-reversal, cannot be made unambiguously. The authors make the following concrete application of this result. According to Yang and Tiomno, a suitable choice of Fermion types makes the beta-decay process (15)

$$n \to p + e^- + \nu \tag{15}$$

(and other observed processes) follow from the existence of a universal coupling between Fermi fields, while the non-existent process

$$n + p \to e^+ + \nu \tag{16}$$

is excluded because the same universal coupling leading to (16) would be non-invariant under time-reversal. The argument of Yang and Tiomno fails because the transformation (14), with $F = \frac{1}{2}\pi N_\lambda$, changes the types of the Fermi fields in such a way as to make the coupling (16) invariant. Thus the absolute conservation of the heavy-particle number N_λ is an empirical fact which cannot be deduced from considerations of invariance under time-reversal.

It is proved that a precisely similar set of conclusions holds for the behavior of fields under space-reflection, the arguments only being slightly simpler because the space-reflection transformation is unitary instead of anti-unitary. *F. J. Dyson* (Princeton, N.J.).

Zimmerman, W. Über den Zusammenhang von Bethe-Salpetergleichung und Tamm-Dancoffmethode. Nuovo Cimento (9) **11** (1954), supplemento, 43–90.

The author investigates in detail the formal connections between the Bethe-Salpeter equation [Phys. Rev. (2) **84** (1951), 1232–1242; MR **14**, 707] and the Tamm-Dancoff method [S. M. Dancoff, ibid. **78** (1950), 382–385]. The results may be adequately summarized by saying that no close or useful correspondence between the two methods is found. Only in the static approximation do the wave-functions in the two methods bear a direct relation to each other. The author analyses the structure of both methods by an extensive use of graphical representations designed for this purpose. *F. J. Dyson.*

Zimmerman, W. Renormierung von Wellenfunktionen in der Feldphysik. Nuovo Cimento (9) **11** (1954), supplemento, 106–117.

It is proved by a detailed analysis that the wave-functions, which appear in the description of 2-particle states in quantum field theory, can be renormalized in the same way as the S-matrix. The proof makes use of the graphical technique developed by the author in the paper reviewed above. *F. J. Dyson* (Princeton, N.J.).

Zimmermann, W. Yang-Feldmanformalismus und einzeitige Wellenfunktionen. Nuovo Cimento (9) **11** (1954), 577–589.

The integral equations of the "New Tamm-Dancoff Method", [F. J. Dyson, Phys. Rev. (2) **91** (1953), 1543–1550; MR **15**, 768], are here derived in a relativistically invariant way from the field-theory formalism of C. N. Yang and D. Feldman [ibid. **79** (1950), 972–978; MR **12**, 569]. The main advantage of this method of derivation is that it resolves some ambiguities caused by the existence of poles in the kernels of the integral equations.

F. J. Dyson (Princeton, N.J.).

Arnous, E. Note on the Tamm-Dancoff method. Proc. Roy. Irish Acad. Sect. A. **57** (1955), 31–36.

Let ψ_λ be the stationary state of a system of n nucleons i.e., $(E_\lambda - H_0 - H)\psi_\lambda = 0$, where H is the interaction Hamiltonian. Write $H = H^- + H^+ + H^=$; H^+, H^- denote those parts of H which contain operators of creation and destruction for mesons, $H^=$ the part which does not change the meson number. The exact solution of the equation above is

$$\psi_\lambda = [1 - [E_\lambda - H_0 - V(E_\lambda)]^{-1}H^+]^{-1}\phi_\lambda^0,$$

where $[E_\lambda - H_0 - V(E_\lambda)]\phi_\lambda^0 = 0$ and

$$V(E_\lambda) = H^- [E - H_0 - V(E)]^{-1}H^+ + H^=.$$

It is remarked, that in this notation, the Tamm-Dancoff approximation corresponds to taking

$$V(E) = H^- P[E - H_0]H^+ + H^=.$$

A. Salam (Cambridge, England).

Schweber, S. S. On the Yang-Feldman formalism. Nuovo Cimento (10) **2** (1955), 397–412.

The chief contributions of this paper are: (1) the establishment, for a typical quantum field theory, of the connection between the field operators in the interaction representation and the "in" and "out" fields of Yang and Feldman [Phys. Rev. (2) **79** (1950), 972–978; MR **12**, 569], all of which satisfy the free-particle equations; and (2) the development of formal techniques based on the Dyson expansion of the interaction transformation operator $U(t_2, t_1)$ [ibid. **75** (1949), 486–502; MR **10**, 418] which enable one to calculate various commutators involving these operations. The methods are used to obtain a closed expression involving only Heisenberg operators, for the S-matrix describing meson-nucleon scattering. As the author points out, this expression together with the "causality" condition: $[\varphi_{\text{in}}(x), \varphi_{\text{out}}(y)] = 0$ for $(x-y)^2 > 0$, leads directly to rigorous proofs (independent of perturbation theory) of the dispersion relations obtained by Gell-Mann, Goldberger, and Thirring [ibid. **95** (1954), 1612–1627; MR **16**, 654; see also M. L. Goldberger, ibid. **97** (1955), 508–510; **99** (1955), 979–985; MR **16**, 1184]. The utility of the Yang-Feldman formalism, which has already been emphasized by Källén [Ark. Fys. **2** (1950), 371–410; Helv. Phys. Acta **26** (1953), 755–760; MR **12**, 890; **15**, 587], is therefore further underscored. In a final paragraph the author treats the "symmetric" field:

$\varphi_{sym}=\frac{1}{2}(\varphi_{in}+\varphi_{out})$, and promises to discuss at a later date the consequences of his results for the reactance matrix K. *B. S. DeWitt* (Chapel Hill, N.C.).

Hara, Osamu, and Shimazu, Haruo. On Yukawa's theory of non-local field. II. The interaction of non-local fields. Progr. Theoret. Physics **9** (1953), 137–146.

[For part I see same journal **7** (1952), 255–262; MR **14**, 117.] Yukawa's non-local field theory [Phys. Rev. (2) **80** (1950), 1047–1052; MR **12**, 571] is here shown to be equivalent to local theory and to give rise to infinite self-energy in second order approximation for a local spinor field interacting with a non-local field even in the limit of vanishing particle radius. *A. J. Coleman.*

Polkinghorne, J. C. Temporally ordered graphs and bound state equations. Proc. Cambridge Philos. Soc. **51** (1955), 762–765.

Using the temporally ordered graphical formalism developed previously by the author [same Proc. **51** (1955), 113–120; MR **16**, 981], a very simple procedure is outlined for writing down the kernels of the single-time three-dimensional bound-state equations, of the type considered by Levy [Phys. Rev. (2) **88** (1952), 72–82; MR **14**, 706], Tamm [Acad. Sci. USSR J. Phys. **9** (1945), 449–460] and Dancoff [Phys. Rev. (2) **78** (1950), 382–385]. *A. Salam* (Cambridge, England).

Polkinghorne, J. C. Normal products of Heisenberg operators. Nuovo Cimento (9) **12** (1954), 317–322.

A new definition is given for the "normal product" of a set of field operators in the Heisenberg representation of quantum field theory. This generalizes the definition which G. C. Wick [Phys. Rev. (2) **80** (1950), 268–272; MR **12**, 380] gave for the case of non-interacting fields. The simplest way to express the new definition is the following. If the field operator $\phi(z)$ is coupled to a c-number source-function $K(z)$, then the normal product satisfies the functional differential equation

$$[\delta/\delta K(z)]N(O(x)O(y)\cdots)=N(\phi(z)O(x)O(y)\cdots).$$

F. J. Dyson (Princeton, N.J.).

Caianiello, E. R. On quantum field theory. II. Non-perturbative equations and methods. Nuovo Cimento (9) **11** (1954), 492–529.

Using the perturbation approach developed in an earlier paper [Nuovo Cimento (9) **10** (1953), 1634–1652; MR **15**, 586], the author defines a kernel function K for an S-matrix element between arbitrary initial and final states. These kernels are shown to satisfy systems of "branching equations" which hold independently of the perturbation methods and may be taken as the axiomatic foundation of the theory. Thus, for example, K_n, the kernel for processes involving $2n$ external fermion lines, satisfies the equation,

$$\frac{d}{d(\lambda^2)}K_n=\left(a+\lambda^2 b\frac{d}{d(\lambda^2)}\right)K_{n+1}.$$

Here λ is the coupling constant and a and b are (complicated) integral operators independent of λ^2. *A. Salam.*

Candlin, D. J. On recurrence relations in field theory. Nuovo Cimento (9) **12** (1954), 380–386.

It is shown that the kernel functions defined by Caianiello in the paper reviewed above are just the vacuum expectation values of T-products of Heisenberg operators. Using the Heisenberg equations of motion, Caianiello's results are more simply rederived. *A. Salam.*

Landau, L., and Pomerančuk, I. On point interaction in quantum electrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 489–492. (Russian)

The asymptotic behavior of the particle propagators was investigated in earlier papers [e.g. L. D. Landau, A. A. Abrikosov and I. M. Halatnikov, same Dokl. (N.S.) **96** (1954), 261–264; MR **16**, 316], discussing a form of quantum electrodynamics with a cut-off acting to suppress momenta higher than a certain limit Λ. A particular result of the earlier work was the relation

$$(1)\qquad e^2=e_1^2[1+(\nu/3\pi)e_1^2\log(\Lambda^2/m^2)]^{-1},$$

between the physical charge e and the "bare charge" e_1. Here ν is the number of distinct charged fields in nature. In this paper it is pointed out that in the limit of a point interaction ($\Lambda\to\infty$), (1) implies $e^2=0$ irrespective of the value of e_1. This is claimed as a proof that quantum electrodynamics with point interactions is mathematically inconsistent. The paper consists mainly of a discussion of the physical consequences of this conclusion. *F. J. Dyson* (Princeton, N.J.).

Galanin, A. D., Ioffe, B. L., and Pomerančuk, I. Ya. On the asymptotics of Green's functions of a nucleon and meson in the pseudo-scalar theory with weak coupling. Ž. Eksper. Teoret. Fiz. **29** (1955), 51–63. (Russian)

The asymptotic behavior of the Green's functions at large momenta is determined for the pseudoscalar theory of meson and nucleon fields. The results agree with those obtained by A. A. Abrikosov, A. D. Galanin and I. M. Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 793–796; MR **16**, 317] and the method of derivation is similar. This presentation differs from the earlier one in two respects: (i) instead of a crude high-momentum cut-off, the exact relativistic theory with renormalization is used throughout; (ii) the writing is less condensed and the details are easier to follow. *F. J. Dyson.*

Nambu, Yoichiro. Structure of Green's functions in quantum field theory. Phys. Rev. (2) **100** (1955), 394–411.

This paper follows another by the same author [Phys. Rev. (2) **98** (1955), 803–811; MR **17**, 333]. The theme of both papers is the generalization to many-particle Green's functions of the formula of H. Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332] for the one-particle Green's function

$$\Delta_F'(x)=(0|P(\varphi(x),\varphi(x'))|0).$$

Here φ represents a scalar field and P a chronological product. Lehmann's formula is

$$\Delta_F'(x)=\int_0^\infty \Delta_m(x)\rho(m^2)dm^2,$$

where $\Delta_m(x)$ is the Feynman propagator for a free field of mass m, and $\rho(m^2)$ is a positive weight-function. The corresponding formulae are explicitly stated for Green's functions derived from products of three and four operator factors. In the 3-point Green's function, Δ_m is replaced by a known invariant function involving 3 scalar parameters u, v, w, and $\rho(m^2)$ is replaced by a weight function $\rho(u, v, w)$. In the 4-point Green's function there is a 6-fold integration over 6 scalar parameters. The determination of the Green's functions is thus reduced to the determination of the "spectral functions" ρ.

In the second half of the paper the field equations are used in order to derive integral equations relating the

various spectral functions ν to one another. From such integral equations the functions may in principle be determined. The equations are automatically renormalized and no trouble arises from divergences. All through the analysis the principle of causality is used in order to determine the analytic behavior of the various functions. The equations themselves are too lengthy to be reproduced in detail here. *F. J. Dyson* (Princeton, N.J.).

Shimose, Tsuneto. On the operational treatment of the Green function formalism. Nat. Sci. Rep. Ochanomizu Univ. **5** (1955), 178–184.

This is mainly a review article. The author reviews the work of Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520], Edwards and Peierls [Proc. Roy. Soc. London. Ser. A. **224** (1954), 24–33; MR **15**, 1010] and others on the construction of the Green functions in the quantum field theory, and he shows how some of the earlier results can be obtained in a somewhat simpler way. The main objective of the Green function formalism is to try to obtain the effect of the radiative corrections on the interaction of particles in a closed form. *S. N. Gupta* (Lafayette, Ind.).

Bogolyubov, N. N., and Širkov, D. V. On the renormalization group in quantum electrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 203–206. (Russian)

The renormalization group is a concept introduced by E. C. G. Stueckelberg and A. Petermann [Helv. Phys. Acta **26** (1953), 499–520]. In its simplest form it is the group of scale transformations

$$D \to zD, \quad e \to z^{-\frac{1}{2}}e, \tag{1}$$

where D is the Maxwell field propagator, e is the electron charge, and z is any finite positive real number. The observable quantities of quantum electrodynamics are invariant under the transformation (1). The authors here use this fact to establish a functional identity satisfied by the function D. Writing

$$D_{\mu\nu}(k)=ik^{-2}[g_{\mu\nu}-k^{-2}k_\mu k_\nu]d(k^2), \tag{2}$$

the normalization of D can be fixed by specifying the value $k^2=\lambda^2$ at which $d(k^2)=1$. Then $d(k^2)$ is rigorously a function of three arguments

$$d(k^2)=d((k^2/\lambda^2), (m^2/\lambda^2), e^2). \tag{3}$$

The transformation (1) implies a simultaneous change in the values of λ and e. From the invariance under this transformation follows the identity

$$d(x, y, e^2)=d(t, y, e^2)d((x/t), (y/t), e^2d(t, y, e^2)). \tag{4}$$

An analogous identity is also proved for the electron propagator.

Eq. (4) is similar to a functional equation proved by M. Gell-Mann and F. Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315], from which asymptotic formulae for the behavior of $d(k^2)$ at high momenta were deduced. The proof in this paper seems to be much simpler and also to assume less than the argument of Gell-Mann and Low. *F. J. Dyson* (Princeton, N.J.).

Bogolyubov, N. N., and Širkov, D. V. Application of the renormalization group to improvement of formulas in perturbation theory. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 391–394. (Russian)

From the identities proved in the paper reviewed above, the authors derive asymptotic relations for the behavior of the propagators in quantum electrodynamics, for both large and small values of the momenta. These relations are by no means sufficient to fix uniquely the forms of the functions. However, if it is assumed that the photon propagator is asymptotically a function of $[e^2 \log (k^2/m^2)]$ alone, then the form of this function is uniquely determined, in agreement with the result of L. D. Landau, A. A. Abrikosov and I. M. Halatnikov. [same Dokl. **96** (1954), 261–264; MR **16**, 316]. The authors do not consider the latter assumption to be mathematically justified, since it rests strongly on perturbation-theory approximations.

There is a misprint in Eq. (13), where the first denominator 4π should be replaced by z. *F. J. Dyson.*

Anderson, James L. Constraints in convariant field theories. II. Phys. Rev. (2) **99** (1955), 1009–1015.

The paper deals with the difficults occurring in the course of quantization of a c-number field theory already put in Hamiltonian form but where constraints are present, i.e. where certain relations exist between field variables and their canonical conjugate, which do not contain the first time derivative of the field variables. Difficulties linked to such c-number theories have been investigated by Dirac [Canad. J. Math. **2** (1950), 129–148; MR **13**, 306], Penfield [Phys. Rev. (2) **84** (1951), 737–743], and Anderson and Bergmann [ibid. **83** (1951), 1018–1025, referred to as I; MR **13**, 411]. The main difficulty is that the constraints in the quantum theory are only defined up to the ordering of factors. It is proved in the paper that there exists at least one ordering for which the constraints plus the Hamiltonian constitute a function group, as in the c-number theory (see I). Then from a c-number theory with constraints a q-number theory may be constructed with the same invariance properties as the c-number theory. The whole proof is based on the existence, in the c-number theory, of a canonical transformation giving to the constraints the simple form

$$\pi^\zeta(x)=0 \quad (\zeta=1, 2, \cdots, n), \tag{1}$$

where π^ζ are the momenta canonically conjugate to the "constraint variables" $\phi_\zeta(x)$, n being the number of constraints. Only formal proofs are given, and no particular case is worked out. *D. Rivier* (Lausanne).

Królikowski, W., and Rzewuski, J. Co-variant one-time formulation of the many-body problem in the quantum theory. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 353–359.

The aim of the authors is to reduce the integro-differential equations in quantum field theory from their original four-dimensional to an equivalent three-dimensional form. This is achieved in principle by means of a general theorem whose proof is deferred to a further paper. The connection between the three- and four-dimensional kernels is rather involved, so the authors suggest a perturbation approach for practical applications. *S. Fubini* (Torino).

Deser, S. Functional integrals and adiabatic limits in field theory. Phys. Rev. (2) **99** (1955), 325–327.

A proof is given of the fact that the behavior of a system of coupled fields, as the masses of some of the particles described by the fields are taken to be greater than the others, tends to be identical with that of a reduced system in which the heavy particles act as static sources of the fields to which they are coupled. The proof uses the propagation functions (renormalized Green's functions) of the

interacting particles expressed as functional integrals, according to Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520]. The following cases are considered: several fermions via boson field of zero mass with scalar, vector, etc. couplings. Special attention is given to couplings which have no simple behavior in the nonrelativistic limit. First, vacuum polarization effects are neglected, then it is shown that there can be no vacuum polarization effects of the fields corresponding to the heavy particles, since they tend to lose their negative-energy components when the particles become very heavy. *D. Rivier* (Lausanne).

Källén, G. The coupling constant in field theory. Nouvo Cimento (9) **12** (1954), 217–225.

This paper originated as a critique of the argument by which S. Deser, W. Thirring and M. L. Goldberger [Phys. Rev. (2) **94** (1954), 711–723] tried to establish a contradiction between pseudoscalar meson field theory and experimental results. By a simple example the author shows that the only conclusion which logically follows from experimental data is that the pseudoscalar theory cannot be adequately tested with weak-coupling approximations. *F. J. Dyson* (Princeton, N.J.).

Schrödinger, E. The wave equation for spin 1 in Hamiltonian form. II. Proc. Roy. Soc. London. Ser. A. **232** (1955), 435–447.

A detailed investigation of the formal properties of the Proca equations for a spin-1 particle, using the notations of the author's earlier paper [same Proc. **229** (1955), 39–43; MR **17**, 219]. It is found that the quantities which obey differential conservation laws (current-density and momentum-density) are not the same as the quantities which commute with the Hamiltonian. The difference between the two ways of defining "conserved quantities" arises from the fact that the formalism possesses an indefinite metric. All the complications belong to the Proca equations considered as a "one-particle theory" and are removed by the process of second quantization. *F. J. Dyson* (Princeton, N.J.).

Case, K. M. Wave equation for spin 0 in Hamiltonian form. Phys. Rev. (2) **99** (1955), 1572–1573.

Schrödinger has shown [Proc. Roy. Soc. London. Ser. A. **229** (1955), 39–43; MR **17**, 219] that the equations for a spin 1 particle can be written in the Hamiltonian form,

$$i\frac{\partial\Psi}{\partial t}=\mathcal{H}\Psi,$$

where $\mathcal{H}=\beta P+K\beta_4$ and the β's form a 10-dimensional representation of the Duffin-Kemmer algebra. The wave function Ψ is subject to the initial condition $(\mathcal{H}\beta_4-K)\Psi=0$. The author demonstrates that the same equations describes spin-zero particles as well if the 5-dimensional representation of the algebra is used. *A. Salam.*

Klein, Abraham. Low-energy theorems for renormalizable field theories. Phys. Rev. (2) **99** (1955), 998–1008.

The paper gives a method for investigating the low-energy behavior of collision processes. The method is to start with the so-called "renormalized Green's functions" for the colliding particles in the presence of external sources, as introduced by Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520]. The collision of a boson with a fermion is treated, with applications: in the first place, to the case of the scattering of γ-rays by a proton, leading to the theorem that the scattering amplitude is given by a Born approximation up to terms of first order in photon energy; in the second place, to the case of meson-nucleon scattering, where a similar theorem (restricted to the P-wave) is established; and finally to the more complex situation of photomeson production in order to prove three theorems: first the Krool-Ruderman theorem [Phys. Rev. (2) **93** (1954), 233–238]; secondly, that the P-wave production from the nucleon is given near the threshold by the Born approximation computed with the actual meson-nucleon coupling constant and magnetic moment; and eventually a similar theorem for the contribution of the meson current. *D. Rivier* (Lausanne).

Román, P. Quantelung des elektromagnetischen Feldes in einer neuartigen Darstellung. Acta Phys. Acad. Sci. Hungar. **4** (1955), 209–218. (Russian summary)

Methods used by Green and Wolf [Proc. Phys. Soc. Sect. A. **66** (1953), 1129–1137] and Novobatzky [Z. Physik **111** (1938), 292–300], applicable only to the pure radiation field, are extended to fields in the presence of finite charged densities. The field equations without the Lorentz condition are used. A canonical formation is developed in terms of functions whose Fourier coefficients are complex combinations of those of the vector potential (longitudinal part eliminated) and the current density. This formalism is formally similar to that of a complex scalar field. Angular momentum is also discussed and the field is quantised. *C. Strachan* (Aberdeen).

Katayama, Y., Tokuoka, Z., and Yamazaki, K. Overall space-time description and third quantization. Nuovo Cimento (10) **2** (1955), 728–744.

The authors discuss a formalism of quantum field theory, which is pre-renormalized and does not require the existence of a Hamiltonian. Their treatment makes use of the notions of Feynman amplitudes and third quantization, and it is based on the earlier work of Nambu [Progr. Theoret. Phys. **4** (1949), 331–346, 399–411; MR **11**, 766], Matthews and Salam [Proc. Roy. Soc. London. Ser. A. **221** (1954), 128–134; MR **15**, 586] and Coester [Phys. Rev. (2) **95** (1954), 1318–1323; MR **16**, 320]. The resulting formalism is far more complicated than the usual formalism of the field theory, and it seems to involve some difficulties of physical interpretation. *S. N. Gupta* (Lafayette, Ind.).

Gupta, Suraj N. Multiple photon production in quantum electrodynamics. Phys. Rev. (2) **98** (1955), 1502–1511.

Multiple production of photons in high-energy processes in quantum electrodynamics is discussed with particular reference to the production when an electron-positron pair annihilates. The cross-section in the centre-of-mass system for n photon production is approximately,

$$\sigma_n=\alpha^n\frac{1}{n-2!}\frac{2^{n-2}}{\pi^{n-3}}\frac{1}{E^2}\left(\log\frac{2E}{m}\right)^{n-1}\left(\log\frac{E}{\varepsilon}\right)^{n-2}.$$

ε is the lowest photon energy observable. It is concluded that at incident positron energies of the order of 10^{14} ev., $\sigma_n\approx\sigma_2/(n-2)!$ and is thus comparable to σ_2 for small n. *A. Salam* (Cambridge, England).

Klein, Abraham, and McCormick, Bruce H. Meson pair theory. Phys. Rev. (2) **98** (1955), 1428–1445.

Meson-pair theory with fixed source is presented with

the use of the Møller scattering matrix [Danske Vid. Selsk. Mat.-Fys. Medd. **23** (1945), no. 1]. The relation of this to the S-matrix treatment is discussed. First the Heisenberg operators for the neutral meson field are derived for a time-independent separable interaction with a nucleon source function. The Møller scattering matrix Λ is constructed, also the S-matrix, both are proved to be unitary, and the self-energy is derived. The Fock-space representation of Λ is derived. The basic elements are the vacuum-to-vacuum amplitude, the matrix giving the proper mesic field of the source and the one-meson to one-meson amplitude. The theory contains no real processes other than meson scattering. Renormalisation is discussed.

In an appendix another treatment of the problem is given using equations of motion of the source [Edwards and Peierls, Proc. Roy. Soc. London. Ser. A. **224** (1954), 24–33; MR **15**, 1010]. *C. Strachan* (Aberdeen).

Medvedev, B. V. On the construction of the scattering matrix in quantum field theory with nonlocal interaction. Dokl. Akad. Nauk. SSSR (N.S.) **103** (1955), 37–40.

Let a non-local field theory be given, for example a theory of a spinor and a scalar field with the interaction

$$\Lambda=F(x, x', x'')\bar{\psi}(x)u(x')\psi(x'').$$

The author proves that it is always possible to construct an S-matrix which has the properties of (i) unitarity, (ii) relativistic invariance, (iii) approximate causality. The notion of approximate causality is carefully defined; it means that uncausal effects can persist only over the short intervals of time between points x, x', x'' at which the form-function F is non-zero. In the construction of the S-matrix, an infinite sequence of supplementary interactions is added to the original interaction Λ.

F. J. Dyson (Princeton, N.J.).

Wigner, Eugene P. Derivative matrix and scattering matrix. Rev. Mexicana Fis. **1** (1952), 91–101.

The discussion of the relation between the scattering matrix S and the matrix R begun by the author in Ann. of Math. (2) **53** (1951), 36–67 [MR **12**, 490] is here continued. He shows that in order to give rise to a permissible R, the matrix S must satisfy the following rule: In an ordering of the purely imaginary poles of S according to their absolute value, there occurs first an even or zero number of negative imaginary poles and thereafter odd numbers of positive and negative poles alternate.

A. J. Coleman (Toronto, Ont.).

Wigner, Eugene P. Derivative matrix and scattering matrix. Rev. Mexicana Fis. **1** (1952), 81–90. (Spanish)

Spanish translation of the paper reviewed above.

Tani, Smio, and Yamazaki, Kazuo. On the meaning of the solution to the one-body Dirac equation. Progr. Theoret. Phys. **12** (1954), 723–746.

First Furry's theorem [Phys. Rev. (2) **81** (1951), 115–124; MR **12**, 785] is proved by means of a canonical transformation applied to field operators, i.e. that in the treatment of fermions interacting with an external field, it is sufficient to know the complete set of solutions of the unquantised one-body Dirac equation in order to predict the results of the quantised theory. This is true also in hole-theory. A similar theorem is not in general true, e.g. for nucleons interacting with neutral pseudoscalar mesons by pseudoscalar coupling, but the distinction between the one-body solution and that of quantised fields is estimated.

The theory is also applied to the hole-theory of fermions; the operator techniques used and correction terms arising from the negative-energy distribution of fermions are discussed. *C. Strachan* (Aberdeen).

Umezawa, H., and Visconti, A. General theory of propagators. Nuovo Cimento (10) **1** (1955), 1079–1103.

An electron-positron field interacting with an electromagnetic field is discussed. A 'source-free' representation is used in which the electron and electromagnetic fields obey the same equations as they would in a Heisenberg representation without sources and whose state vectors are assumed to be related to those of the Heisenberg representation by a unitary transformation U and to agree with them on a specified surface. The sources influence the state vector only, not the field operators. By means of the functional derivatives of U, propagators may be defined in terms of which there is a functional Taylor expansion of U. Graphical and physical interpretations are given. Equations determining all possible propagators may be derived and the theory is applied to one-electron, one-photon, and two-electron propagators. Equations for Compton and Møller scattering different from those of Bethe-Salpeter are derived and make the renormalisation procedure automatic. The solution of these equations is discussed together with the relation of the theory to conventional renormalisation.

C. Strachan (Aberdeen).

Bass, L., and Schrödinger, E. Must the photon mass be zero? Proc. Roy. Soc. London. Ser. A. **232** (1955), 1–6.

Les auteurs rappellent que l'hypothèse d'une masse non nulle, si petite soit-elle, entraîne l'existence d'ondes longitudinales, et que si ces dernières se comportaient de manière analogue aux ondes tranversales, en particulier pour les effets calorifiques et dynamiques de la radiation du corps noir, certaines constantes thermodynamiques, comme la constante de Stefan, devraient à première vue tout au moins être multipliées par un facteur 3/2. En considérant: d'une part le champ électromagnétique comme un cas limite d'un champ de Proca gouverné par des équations de Maxwell généralisées

$$\operatorname{curl} E+\dot{H}=0, \qquad \operatorname{curl} H-\dot{E}=-\mu A,$$
$$\operatorname{div} H=0, \qquad \operatorname{div} E=-\mu V,$$
$$\operatorname{curl} A=\mu H,$$
$$\operatorname{grad} V+\dot{A}=-\mu E, \quad \operatorname{div} A+\dot{V}=0,$$

où le champ (E, H) est faiblement couplé avec le quadrivecteur (A, V) par l'intermédiaire du coefficient μ proportionnel à la masse du photon; et d'autre part une cavité limitée par des parois parfaitement conductrices, le paradoxe est levé dans ce cas, en ce sens que, à l'opposé des ondes tranversales qui sont presque parfaitement réfléchies par les parois, les ondes longitudinales sont presque parfaitement transmisses dans les parois. Un calcul élémentaire montre que la perte d'énergie rayonnante (des ondes transversales) consécutive à la présence des ondes longitudinales ne devient appréciable qu'après un intervalle de temps $\Delta t>\Omega\frac{1}{3}\cdot 3\cdot 10^{17}$ secondes, Ω étant le volume de la cavité mesuré en cm. Les raisonnements sont classiques (sans quantification). *D. Rivier.*

Urbanik, K. Bemerkungen über die mittlere Anzahl von Partikeln in gewissen stochastischen Schauern. Studia Math. **15** (1955), 34–42.

The author considers first nucleon (=proton or neutron) cascades. Denote by $m(\varepsilon, x)$ the expected number of particles at depth x with energy exceeding εE_0, where E_0 is the energy of the primary particle. Then $\lim_{x\to\infty} e^{\lambda x} m(\varepsilon, x) = 0$ if $\lambda < \alpha$, but $= \infty$ if $\lambda \geqq \alpha$, where α is the total cross-section of the scattering. A similar formula is derived for electron-photon cascades. *E. Lukacs.*

Dirac, P. A. M. Note on the use of non-orthogonal wave functions in perturbation calculations. Canad. J. Phys. **33** (1955), 709–712.

Certain collision problems involve initial and final states which do not belong to the same orthogonal set of wave functions. As an example the author considers the transfer of an electron from atom A to ion B^+ resulting in ion A^+ and atom B. The Hamiltonian H is $H(A)+H(B^+)+V_1$ before and $H(A^+)+H(B)+V_2$ after the transfer. The interactions V_1 and V_2 are different functions. Let ϕ_λ and χ_μ denote the two orthogonal sets of eigenfunctions of H in the cases of $V_1=0$ and $V_2=0$, respectively. The initial state ψ_0 belongs to ϕ_λ and is not orthogonal to the χ_μ. The two sets are not complete and, in general, neither is their sum. However, provided the ϕ_λ and the χ_μ are independent, the projection of a function onto the sum space is uniquely determined. It is now asserted that in a first approximation the function $V_1\psi_0$ is given by its projection onto that space. An explicit solution in first-order stationary-state perturbation theory can then be obtained. The difficulty of proving the crucial independence of the ϕ_λ and the χ_μ is pointed out and it is proposed, if necessary, to obtain this independence by working with suitable subsets, which, however, would make the approximation worse. *F. Rohrlich.*

Olsen, Haakon. Out and in-going waves in final states. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 10–15.

In the theory of bremsstrahlung matrix elements have to be calculated between stationary scattering states. It is known [G. Breit and H. A. Bethe, Phys. Rev. (2) **93** (1954), 888–890; MR **15**, 919] that the final state has to be chosen as a plane wave plus ingoing waves. It is here shown, however, that for the total cross-section the particular choice of final state is immaterial, at least for particles spin 0 or $\frac{1}{2}$. *N. G. van Kampen* (Utrecht).

Gustafson, Torsten. On the potential collective flow of a rotating nucleus with non-ellipsoidal boundary. Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 5, 16 pp.

This is an attempt to explain that the experimental nuclear moments of inertia are three to five times larger than those computed from the collective nuclear model [summarized by A. Bohr and B. R. Mottelson in Ch. XVII of K. Siegbahn (ed.), Beta- and gamma-ray spectroscopy, North-Holland Publ. Co., Amsterdam, 1955], provided one wants to account for the quadrupole moments Q_0. This model assumes the nucleus to be an irrotational, incompressible fluid of spheroidal shape. The author tries to account for the discrepancy by deviations from this shape. Using prolate spheroidal coordinates he solves Laplace's equation of the velocity potential Φ for moderate deviations from spheroidal shape leaving the volume constant and computing the first three terms of the expansion of Φ in even harmonic functions. He finds that appreciable changes of the moment of inertia $\mathfrak{J}$ can be obtained, but that they are accompanied by changes in Q_0 such that $Q_0^2/\mathfrak{J}$ changes only insignificantly. Thus, fitting Q_0 he cannot account for the experimental $\mathfrak{J}$. *F. Rohrlich* (Iowa City, Ia.).

BIBLIOGRAPHICAL NOTES

Conference on probability theory and its applications, Leningrad, 30 May–4 June, 1955.

A brief description and a list of the papers read is given in Vestnik Leningrad. Univ. **10** (1955), no. 11, 43–45. A number of the individual papers, some only as abstracts, are printed on pp. 46–97. Those which are not abstracts will be reviewed separately.

Wissenschaftliche Tagung der Gesellschaft für angewandte Mathematik und Mechanik in Berlin.

The abstracts of the lectures given at this meeting, held from 31 May to 4 June 1955, are printed in Z. Angew. Math. Mech. **35** (1955), 325–396. Only the Hauptaufsätze will be reviewed in MR.

Proceedings of the Eighth International Congress on Theoretical and Applied Mechanics, Istanbul, August 20-28, 1952. Part I. Faculty of Science, University of Istanbul, 1953 [1955]. xi+530 pp. $11.00.

This volume contains the list of members of the congress, the program, opening adresses, and abstracts of presented papers.

Publications Scientifiques de l'Université d'Alger. Série A. Sciences Mathématiques.

Vol. 1, no. 1, is dated June 1954, but did not appear until 1955. Correspondence and subscriptions should be addressed to B. d'Orgeval, Faculté des Sciences, 2 rue Michelet, Algiers.

Revista Universităţii „C. I. Parhon" şi a Politehnicii Bucureşti. Seria Ştiinţelor Naturii.

No. 1 of this journal is dated 1952, nos. 2 and 3, 1953, nos. 4–5, 1954, and nos. 6–7 and 8, 1955. The address is: Universitatea C. I. Parhon, B-dul 6 Martie, Nr. 64, Bucureşti. Papers are in Romanian with summaries in Russian and French.

Stroje na Zpracování Informací.

Sbornik I of this journal is dated 1953, Sbornik II, 1954. It is a publication of the Laboratory of Mathematical Machines of the Czechoslovak Academy of Sciences (Československá Akademie Věd, Laboratoř Matematických Strojů). The title means "Machines for Processing Information".

Tokyo Kyoiku Daigaku. Sûgaku Kenkyu Roku.

This is a publication of the Tokyo University of Education (Tokyo Kyoiku Daigaku). Vol. 1, no. 1, is dated 1950. The following numbers have been published: vol. 1, nos. 1, 2–3, 4, 5–6, 7, 8–9; vol. 2, nos. 1, 2–3, 4–5, 6–7, 8, 9. Many of the papers are in Japanese and many appear later in other journals in a more widely read language.